

# Syntax and semantics of focalisation with relative monads and comonads

Éléonore Mangel *IRIF, Paris*

Paul-André Melliès *IRIF, INRIA, Paris*

Guillaume Munch-Maccagnoni *INRIA, LS2N CNRS, Nantes*

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## Abstract

The logical principles of focalisation and polarisation can be used to design well-behaved term syntaxes for sequent calculus, which play a role as meta-languages for describing effectful computation. On the semantics side, this corresponds to an axiomatic and polarised notion of model of computation stated in terms of non-associative categories as well as adjunctions between “bare” functors (reflexive graph morphisms) over such non-associative categories.

In this paper, we study the special and delicate cases of resource and effect modalities in a general intuitionistic and linear setting: an exponential comonad  $!$  (refining the necessity modality  $\Box$ ) and a strong monad (written  $\Diamond$ ). The starting point of our contribution is noticing that the completeness for a polarised syntax for  $!$  and  $\Diamond$  with respect to (co)monads in linear call-by-push-value models can be achieved if we move to *relative* (co)monads (Altenkirch, Chapman and Uustalu, 2015; Arkor and McDermott, 2024): more precisely, comonads relative to  $\downarrow$  (the positive shift functor) for  $!$  and monads relative to  $\uparrow$  (the negative shift functor) for  $\Diamond$ .

These specialisations of the concept of relative (co)monad to call-by-push-value adjunctions recently appeared in Jiang, Xue and New (2025) and Melliès (2025, 2026). Yet the syntax we present arose from proof-theoretic consideration in Munch-Maccagnoni (2009) and Curien, Fiore and Munch-Maccagnoni (CFMM 2016), without the link with relative (co)monads being noticed at the time. Our first remark and explanation is thus that (co)monads relative to a call-by-push-value adjunction have been motivated previously from a proof-theoretic perspective in the context of focalisation, which also provides a meta-language for these concepts in an effectful setting.

We carry out the study of these modalities from the axiomatic, non-associative point of view. We recall the definition of adjunction between bare functors in this context, and establish correspondence results between this notion of adjunction and that of relative adjunction. This correspondence is then extended to linear-non-linear and strong versions of adjunctions as needed to model  $!$  and  $\Diamond$ .

## 1. From focalisation and call-by-push-value to polarised calculi

In this paper, we are interested in the links between the semantics of effectful programs featuring linearity, and the proof theory of intuitionistic multiplicative-additive linear sequent calculus (IMALL, see Figure 1). On the syntactic side we consider a polarised version of IMALL,

|                                                                                                                                                                                                                                                                      |                                                                                      |                                                                                                                                                                                                                                                                                                                     |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| polarities                                                                                                                                                                                                                                                           | $\varepsilon ::= + \mid -$                                                           |                                                                                                                                                                                                                                                                                                                     |
| types                                                                                                                                                                                                                                                                | $A, B, A_\varepsilon ::= P \mid N$                                                   |                                                                                                                                                                                                                                                                                                                     |
| positive                                                                                                                                                                                                                                                             | $P, Q, A_+ ::= X^+ \mid \mathbf{1} \mid A \otimes B \mid A \oplus B \mid \mathbf{0}$ | <ul style="list-style-type: none"> <li>Judgements are of the form <math>\Gamma \vdash A</math></li> <li><math>\Gamma</math> is an ordered list of formulæ <math>A_1, \dots, A_n</math>.</li> <li><math>\Gamma, \Gamma'</math> denotes the concatenation of <math>\Gamma</math> and <math>\Gamma'</math>.</li> </ul> |
| negative                                                                                                                                                                                                                                                             | $N, M, A_- ::= X^- \mid A \multimap B \mid A \& B \mid \top$                         | (b) Judgements                                                                                                                                                                                                                                                                                                      |
| (a) Types                                                                                                                                                                                                                                                            |                                                                                      |                                                                                                                                                                                                                                                                                                                     |
| $\frac{}{A \vdash A} \text{ (ax)} \quad \frac{\Gamma \vdash A \quad \Gamma', A \vdash B}{\Gamma, \Gamma' \vdash B} \text{ (cut)} \quad \frac{\Gamma, A, B, \Gamma' \vdash C}{\Gamma, B, A, \Gamma' \vdash C} \text{ (ex)}$                                           |                                                                                      |                                                                                                                                                                                                                                                                                                                     |
| (c) Identity                                                                                                                                                                                                                                                         |                                                                                      |                                                                                                                                                                                                                                                                                                                     |
| (d) Structure                                                                                                                                                                                                                                                        |                                                                                      |                                                                                                                                                                                                                                                                                                                     |
| $\frac{\Gamma \vdash A \quad \Gamma', B \vdash C}{\Gamma, \Gamma', A \multimap B \vdash C} (\multimap \vdash) \quad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B} (\vdash \multimap) \quad \frac{}{\vdash \mathbf{1}} (\vdash \mathbf{1})$                  |                                                                                      |                                                                                                                                                                                                                                                                                                                     |
| $\frac{\Gamma \vdash A \quad \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \otimes B} (\vdash \otimes) \quad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C} (\otimes \vdash) \quad \frac{\Gamma \vdash C}{\Gamma, \mathbf{1} \vdash C} (\mathbf{1} \vdash)$ |                                                                                      |                                                                                                                                                                                                                                                                                                                     |
| $\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \& B} (\vdash \&) \quad \frac{\Gamma, A_{i \in \{1,2\}} \vdash C}{\Gamma, A_1 \& A_2 \vdash C} (\&_i \vdash_f) \quad \frac{}{\Gamma \vdash \top} (\vdash \top)$                                        |                                                                                      |                                                                                                                                                                                                                                                                                                                     |
| $\frac{\Gamma, A \vdash C \quad \Gamma, B \vdash C}{\Gamma, A \oplus B \vdash C} (\oplus \vdash) \quad \frac{\Gamma \vdash A_{i \in \{1,2\}}}{\Gamma \vdash A_1 \oplus A_2} (\vdash \oplus_i) \quad \frac{}{\Gamma, \mathbf{0} \vdash A} (\mathbf{0} \vdash)$        |                                                                                      |                                                                                                                                                                                                                                                                                                                     |
| (e) Logic                                                                                                                                                                                                                                                            |                                                                                      |                                                                                                                                                                                                                                                                                                                     |

Figure 1: **IMALL**

more precisely a linear and intuitionistic version of polarised classical logic described by Girard (1991) and Danos, Joinet and Schellinx (DJS 1997), which originated in a Curry-Howard interpretation of focusing proof search (Andreoli, 1992) with roots in linear logic (Girard, 1987). Systems arising from this perspective are *polarised* in the sense that a difference between positive and negative formulæ appears and is handled formally. This distinction between formulæ is similar to the one type segregation in the call-by-push-value model of programming effects (CBPV: Levy 1999, 2004).

On the semantic side, we consider a linear version of CBPV, the linear CBPV models (Fiore, 2011; Melliès, 2012b; CFMM 2016). Recall that linear CBPV models are enriched (or strong) adjunctions between a (symmetric monoidal and distributive) category of values and a category of co-values (cartesian closed relatively to  $\mathcal{V}$  in a certain sense):

$$\begin{array}{ccc}
 & \uparrow & \\
 \mathcal{V} & \xrightarrow{\quad} & \mathcal{S} \\
 & \downarrow & \\
 & \perp & 
 \end{array}
 \quad (1)$$

The interest in such situations arose after a long series of works starting from monads used to model side-effects (Moggi, 1989, 1991), which soon after shifted attention to adjoint situations decomposing monads as pictured above (Fiore, 1994; Power and Robinson, 1997; Thielecke, 1997; Levy, 1999, 2004, 2005),<sup>1</sup> followed by various attempts to “linearise” it, including the aforemen-

<sup>1</sup>As a matter of fact the works on polarisation by Girard (1991, 1993) can themselves be considered historically

tioned works and others such as Egger, Møgelberg and Simpson (2009).

We avoid developing the full details of strength for the adjunction above. Being enriched (or strong) for the situation (1) means that additional structure is needed to express multiple formulæ on the left-hand side of sequents, obtained by asking the above adjunction to underlie a  $\widehat{\mathcal{V}}$ -enriched adjunction where  $\widehat{\mathcal{V}}$  is the symmetric monoidal category of presheaves on  $\mathcal{V}$  (CFMM 2016), or equivalently by asking that the adjoints are compatible with the monoidal product of  $\mathcal{V}$  and with the action of  $\mathcal{V}^{\text{op}}$  on  $\mathcal{S}$  in a manner defined in Melliès (2012b).

As is by now well understood, a similar situation as (1) underpins Girard’s polarised logics (Girard, 1991, 1993), albeit in a manner which is both more specific—by pertaining to (constructive) classical, intuitionistic and linear logics: said differently to non-linear, linearly-used and fully linear continuations (respectively)—and more general: for instance, they integrate in the case of Girard (1993) several adjunctions in the picture above as opposed to a single one. Many people should be cited for establishing the links between polarised logics and CBPV, mainly via the relationships they have in common with negative translations, and continuation-passing style.<sup>2</sup>

But a strength of the proof-theoretical point of view symbolised by polarised logics is that they can be a source of inspiration for better understanding CBPV models. In this paper, we would like indeed to further understand such situations used in programming language semantics, particularly when comes the consideration of a comonad on  $\mathcal{V}$  and/or a monad on  $\mathcal{S}$  which are again strong in a suitable sense—these are important situations related to the study of resources and effects as modalities. Thus we find it useful to start by recalling the links between effectful semantics and polarised logics.

## 1.1. The deductive system associated to an adjunction

Proof theory is concerned with the formal study of the structure of deductive systems. Categorical proof theory has historically approached deductive systems as categories, given by:

1. a collection of objects (formulæ),
2. for each pair of objects  $A, B$  a set of morphisms from  $A$  to  $B$  (of proofs  $A \vdash B$ ),
3. a well-typed and associative composition operation ( $A \vdash B$  and  $B \vdash C$  give  $A \vdash C$ ), and
4. for each object  $A$  an identity morphism from  $A$  to  $A$  ( $A \vdash A$ ), neutral for composition.

A useful perspective brought by proof theory is that an adjoint situation such as (1) can indeed be analysed as a deductive system, however with a composition which is not associative a priori. Given any (associative) adjoint situation

$$L : \mathcal{A} \rightleftarrows \mathcal{B} : R$$

(where this notation denotes that  $R : \mathcal{B} \rightarrow \mathcal{A}$  is right adjoint to  $L : \mathcal{A} \rightarrow \mathcal{B}$ ), one can define a deductive system  $\mathbf{dupl}_{L,R}$  as follows:

$$|\mathbf{dupl}_{L,R}| = |\mathcal{A}| \uplus |\mathcal{B}| \qquad \mathbf{dupl}_{L,R}(A, B) = \mathcal{O}(A^+, B^-)$$

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as a source of inspiration for the adjoint point of view, via Benton and Wadler (1996) and Streicher and Reus (1998).

<sup>2</sup>for instance Murthy, 1992; Lafont, Reus and Streicher, 1993; Thielecke, 1997; Streicher and Reus, 1998; Ogata, 2000; Curien and Herbelin, 2000; Selinger, 2001; Ogata, 2002; Hofmann and Streicher, 2002; Laurent, 2002, 2003; Laurent, Quatrini and Tortora de Falco, 2005; Melliès and Tabareau, 2010.

where

$$\mathcal{O} : \mathcal{A}^{\text{op}} \times \mathcal{B} \rightarrow \text{Set}$$

is a chosen distributor of oblique morphisms of the adjunction, and where by definition

$$X^+ \stackrel{\text{def}}{=} \begin{cases} X & \text{if } X \in |\mathcal{A}| \\ RX & \text{if } X \in |\mathcal{B}| \end{cases} \quad X^- \stackrel{\text{def}}{=} \begin{cases} LX & \text{if } X \in |\mathcal{A}| \\ X & \text{if } X \in |\mathcal{B}| \end{cases}$$

Objects in  $\mathcal{A}$  are called *positive* in  $\mathbf{dupl}_{L,R}$  and those in  $\mathcal{B}$  *negative*. It is an interesting exercise to check that for all objects  $A, B, C$  one can define a composition

$$\mathbf{dupl}_{L,R}(C, B) \times \mathbf{dupl}_{L,R}(A, C) \rightarrow \mathbf{dupl}_{L,R}(A, B)$$

and that this composition is associative if and only if the adjunction  $L : \mathcal{A} \rightleftarrows \mathcal{B} : R$  is idempotent, in the sense of its multiplication or co-multiplication being iso (Munch-Maccagnoni, 2014).

**Non-associative categories**, such as  $\mathbf{dupl}_{L,R}$ , are simply categories without the assumption that composition is associative. We use the notation  $\circ$  for composition operations that are not associative a priori.

Additional structure, such as the symmetric monoidal and distributive structure on  $\mathcal{V}$  and the cartesian closed structure on  $\mathcal{S}$  relative to  $\mathcal{V}$  (1), allows us to interpret logical connectives. In fact:

**Proposition 1** (CFMM 2016). *The deductive system  $\mathbf{dupl}_{\uparrow, \downarrow}$  associated to a linear CBPV adjunction (1) satisfies the rules of IMALL (Figure 1).*

From the point of view of deductive systems as non-associative categories, just as in categorical proof theory in general, the interpretation of logics into the models determine a notion of equality between derivations which we seek to understand and respect (e.g. during cut elimination). This is the point of view of polarised logics (Girard, 1991), which can still be described and analysed using the tools of category theory, motivating a theory of non-associative categories developed in Munch-Maccagnoni (2014) and Mangel, Melliès and Munch-Maccagnoni (MMMM 2026) and which we partially recall and extend in this paper.

## 1.2. The calculus $\mathbf{IMALL}_p$ : A polarised syntax for linear CBPV

$\mathbf{IMALL}_p$  (**polarised IMALL**) is a term calculus that can be seen as a linear and direct-style version of Levy's CBPV (Levy, 2004). We recall and adapt its definition from CFMM 2016 in Figure 2 and 3.

The calculus  $\mathbf{IMALL}_p$  is defined à la Barendregt, which is to say that we start with an untyped syntax of pseudo-terms à la Curry (Figure 2). Out of these pseudo-terms, the (proper) terms are those that possess a valid typing derivation in the sense of Figure 3. The equational theory  $\simeq_{\text{RE}}$  is obtained as the well-typed restriction of the compatible equivalence closure of reductions  $\triangleright_R$  (Figure 2b,  $\beta$ -like) and expansions  $\triangleright_E$  (Figure 2c,  $\eta$ -like). The calculus  $\mathbf{IMALL}_p$  enjoys the good properties that follow (for more details see CFMM 2016; Munch-Maccagnoni, 2017).

From the point of view of typing,  $\mathbf{IMALL}_p$  expresses the logic  $\mathbf{IMALL}$ , which is evident from Figure 3 and 4. The calculus  $\mathbf{IMALL}_p$  and its conversion rules can be interpreted in any linear CBPV model along the lines of the proof-theoretic interpretation of Proposition 1. The interpretation into linear CBPV models is sound: any two terms related by  $\simeq_{\text{RE}}$  have the same interpretation in all models.

On the proof-theoretic side, the compatible closure  $\rightarrow_R$  of reduction  $\triangleright_R$  is confluent. This is due to the fact that the reduction relation  $\triangleright_R$  defines an orthogonal higher-order rewriting

|                                 | 1                                                                                                                                                         | $\otimes$                                                                    | $\oplus (i \in \{1,2\})$     | $\multimap$                                                                                                                  | $\& (i \in \{1,2\})$ | $\top/0$ |
|---------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------|------------------------------|------------------------------------------------------------------------------------------------------------------------------|----------------------|----------|
| Values:                         |                                                                                                                                                           |                                                                              |                              |                                                                                                                              |                      |          |
|                                 | $V, W ::= x \mid \mu\alpha^-.c \mid () \mid V \otimes W \mid \iota_i(V) \mid \mu(x.\alpha).c \mid \mu<\alpha.c; \beta.c'> \mid \mu<V>$                    |                                                                              |                              |                                                                                                                              |                      |          |
| Stacks:                         |                                                                                                                                                           |                                                                              |                              |                                                                                                                              |                      |          |
|                                 | $S ::= \alpha \mid \bar{\mu}x^+.c \mid \bar{\mu}().c \mid \bar{\mu}(x \otimes y).c \mid \bar{\mu}[x.c \mid y.c'] \mid V.S \mid \pi_i.S \mid \bar{\mu}[S]$ |                                                                              |                              |                                                                                                                              |                      |          |
| Expressions:                    | Contexts:                                                                                                                                                 | Commands:                                                                    |                              |                                                                                                                              |                      |          |
| $t, u ::= V \mid \mu\alpha^+.c$ | $e ::= S \mid \bar{\mu}x^-.c$                                                                                                                             | $c ::= \langle V \parallel e \rangle^- \mid \langle t \parallel S \rangle^+$ |                              |                                                                                                                              |                      |          |
| (a) Terms                       |                                                                                                                                                           |                                                                              |                              |                                                                                                                              |                      |          |
| $(R\bar{\mu}^\varepsilon) :$    | $\langle V \parallel \bar{\mu}x^\varepsilon.c \rangle^\varepsilon$                                                                                        | $\triangleright_R c[V/x]$                                                    | $(E\bar{\mu}^\varepsilon) :$ | $e \triangleright_E \bar{\mu}x^\varepsilon.\langle x \parallel e \rangle^\varepsilon$                                        |                      |          |
| $(R\mu^\varepsilon) :$          | $\langle \mu\alpha^\varepsilon.c \parallel S \rangle^\varepsilon$                                                                                         | $\triangleright_R c[S/\alpha]$                                               | $(E\mu^\varepsilon) :$       | $t \triangleright_E \mu\alpha^\varepsilon.\langle t \parallel \alpha \rangle^\varepsilon$                                    |                      |          |
| $(R1) :$                        | $\langle () \parallel \bar{\mu}().c \rangle^+$                                                                                                            | $\triangleright_R c$                                                         | $(E1) :$                     | $S \triangleright_E \bar{\mu}().\langle () \parallel S \rangle^+$                                                            |                      |          |
| $(R\otimes) :$                  | $\langle V \otimes W \parallel \bar{\mu}(x \otimes y).c \rangle^+$                                                                                        | $\triangleright_R c[V/x, W/y]$                                               | $(E\otimes) :$               | $S \triangleright_E \bar{\mu}(x \otimes y).\langle x \otimes y \parallel S \rangle^+$                                        |                      |          |
| $(R\oplus) :$                   | $\langle \iota_i(V) \parallel \bar{\mu}[x_1.c_1 \mid x_2.c_2] \rangle^+$                                                                                  | $\triangleright_R c_i[V/x_i]$                                                | $(E\oplus) :$                | $S \triangleright_E \bar{\mu}[x.\iota_1(x) \parallel S]^+ \mid y.\langle \iota_2(y) \parallel S \rangle^+$                   |                      |          |
| $(R\multimap) :$                | $\langle \mu(x.\alpha).c \parallel V.S \rangle^-$                                                                                                         | $\triangleright_R c[V/x, S/\alpha]$                                          | $(E\multimap) :$             | $V \triangleright_E \mu(x.\alpha).\langle V \parallel x.\alpha \rangle^-$                                                    |                      |          |
| $(R\&) :$                       | $\langle \mu<\alpha_1.c_1; \alpha_2.c_2> \parallel \pi_i.S \rangle^-$                                                                                     | $\triangleright_R c_i[S/\alpha_i]$                                           | $(E\&) :$                    | $V \triangleright_E \mu<\alpha.\langle V \parallel \pi_1.\alpha \rangle^-; \beta.\langle V \parallel \pi_2.\beta \rangle^->$ |                      |          |
| (no rules $R\top, R0$ )         |                                                                                                                                                           |                                                                              | $(E\top) :$                  | $V \triangleright_E \mu<x_1 \otimes \dots \otimes x_n>$                                                                      |                      |          |
|                                 |                                                                                                                                                           |                                                                              | $(E0) :$                     | $S \triangleright_E \bar{\mu}[x_1 \dots x_n.\alpha]$                                                                         |                      |          |
| (b) Reduction rules             |                                                                                                                                                           |                                                                              | (c) Extensionality rules     |                                                                                                                              |                      |          |

Figure 2: **IMALL<sub>p</sub>**: calculus

system (left-linear, no critical pairs). In fact,  $\rightarrow_R$  defines a noetherian (confluent, normalising) and semantics-preserving cut-elimination strategy for **IMALL**. Furthermore integrating expansions into the cut-elimination results, the  $\rightarrow_R$ -normal and  $\rightarrow_E$ -long terms are in correspondence with focused normal forms, from which follows the completeness of focusing proof search (see Appendix E). In this sense, **IMALL<sub>p</sub>** realises a Curry-Howard interpretation of focusing in correspondence with linear CBPV.

### 1.3. An explanation of **IMALL<sub>p</sub>** from its proof-theoretic origins

While these good properties make of **IMALL<sub>p</sub>** a syntax for linear CBPV, they say very little about how it has been designed. Indeed, it has not been designed initially with the goal of being a good syntax for linear CBPV. To explain the origins of **IMALL<sub>p</sub>**, we need to go back to a different work on polarisation than Girard's, to Danos, Joinet and Schellinx's work on proof-relevant **LK** (DJS 1997). In this work, DJS investigated various ways of endowing **LK** with a noetherian (terminating and confluent) cut-elimination procedure by means of translations into linear logic. This cut-elimination procedure is also semantics-preserving by construction, for the semantics determined by the translation into linear logic. With the system **LK<sub>p</sub><sup>η</sup>**, DJS have found a canonical system, in fact related to Girard's original polarised classical sequent calculus **LC**: canonical in the sense that its translation into linear logic uses the least number of modalities ! and ?, and that other known classical systems can be expressed in it.

- Judgements are:

$$\Gamma \vdash t : A \mid \Gamma \mid e : A \vdash \Delta \quad c : (\Gamma \vdash \Delta)$$

- $\Gamma$  is a map from a finite set of variables to types provided with a total order  $\leq_\Gamma$  on its domain, notation  $\Gamma = (x_1 : A_1, \dots, x_n : A_n)$ .
- $\Delta$  is a pair  $\alpha : A$  of a co-variable and a type.

- Concatenation  $(\Gamma, \Gamma')$  is defined when the domains of  $\Gamma$  and  $\Gamma'$  are disjoint.

(a) Judgements

$$\frac{}{x : A \vdash x : A \mid} (\vdash \text{ax}) \quad \frac{}{\mid \alpha : A \vdash \alpha : A} (\text{ax} \vdash)$$

$$\frac{c : (\Gamma, x : A_\varepsilon \vdash \Delta)}{\Gamma \mid \bar{\mu}x^\varepsilon.c : A_\varepsilon \vdash \Delta} (\bar{\mu}^\varepsilon \vdash) \quad \frac{c : (\Gamma \vdash \alpha : A_\varepsilon)}{\Gamma \vdash \mu\alpha^\varepsilon.c : A_\varepsilon \mid} (\vdash \mu^\varepsilon)$$

$$\frac{\Gamma \vdash t : A_\varepsilon \mid \Gamma' \mid e : A_\varepsilon \vdash \Delta}{\langle t \parallel e \rangle^\varepsilon : (\Gamma, \Gamma' \vdash \Delta)} (\text{cut}^\varepsilon)$$

(b) Identity

- $\Sigma(\Gamma; \Gamma')$  is the set of maps  $\sigma : \text{dom } \Gamma \rightarrow \text{dom } \Gamma'$  satisfying  $\Gamma'(\sigma(x)) = \Gamma(x)$  for all  $x \in \text{dom } \Gamma$ .  $\Sigma^*(\Gamma; \Gamma')$  is the subset of  $\Sigma(\Gamma; \Gamma')$  of maps that are bijective.
- The structural rules for **IMALL**<sub>p</sub> are obtained by taking  $\sigma \in \Sigma^*(\Gamma; \Gamma')$  below (renaming, exchange). By taking  $\Sigma(\Gamma; \Gamma')$  for the range of structural maps (adding weakening and contraction) we obtain **LJ**<sub>p</sub>, in correspondence with call-by-push-value models ([CFMM 2016](#)).

$$\frac{\Gamma \vdash t : A \mid}{\Gamma' \vdash t[\sigma] : A \mid} (\vdash \sigma) \quad \frac{\Gamma \mid e : A \vdash \Delta}{\Gamma' \mid e[\sigma] : A \vdash \Delta} (\sigma \vdash) \quad \frac{c : (\Gamma \vdash \Delta)}{c[\sigma] : (\Gamma' \vdash \Delta)} (\sigma)$$

(c) Structure

$$\frac{\Gamma \vdash V : A \mid \Gamma' \mid S : B \vdash \Delta}{\Gamma, \Gamma' \mid V \cdot S : A \multimap B \vdash \Delta} (\multimap \vdash_f) \quad \frac{c : (\Gamma, x : A \vdash \alpha : B)}{\Gamma \vdash \mu(x \cdot \alpha).c : A \multimap B \mid} (\vdash \multimap)$$

$$\frac{\Gamma \vdash V : A \mid \Gamma' \vdash W : B \mid}{\Gamma, \Gamma' \vdash V \otimes W : A \otimes B \mid} (\vdash_f \otimes) \quad \frac{c : (\Gamma, x : A, y : B \vdash \Delta)}{\Gamma \mid \bar{\mu}(x \otimes y).c : A \otimes B \vdash \Delta} (\otimes \vdash)$$

$$\frac{c : (\Gamma \vdash \alpha : A) \quad c' : (\Gamma \vdash \beta : B)}{\Gamma \vdash \mu < \alpha.c; \beta.c' > : A \& B \mid} (\vdash \&) \quad \frac{\Gamma \mid S : A_i \vdash \Delta}{\Gamma \mid \pi_i \cdot S : A_1 \& A_2 \vdash \Delta} (\&_i \vdash_f)$$

$$\frac{c : (\Gamma, x : A \vdash \Delta) \quad c' : (\Gamma, y : B \vdash \Delta)}{\Gamma \mid \bar{\mu}[x.c \mid y.c'] : A \oplus B \vdash \Delta} (\oplus \vdash) \quad \frac{\Gamma \vdash V : A_i \mid}{\Gamma \vdash \iota_i(V) : A_1 \oplus A_2 \mid} (\vdash_f \oplus_i)$$

$$\frac{}{\vdash () : \mathbf{1} \mid} (\vdash \mathbf{1}) \quad \frac{c : (\Gamma \vdash \Delta)}{\Gamma \mid \bar{\mu}().c : \mathbf{1} \vdash \Delta} (\mathbf{1} \vdash) \quad \frac{\Gamma \vdash V : A \mid}{\Gamma \vdash \mu < V > : \top \mid} (\vdash_f \top) \quad \frac{\Gamma \mid S : A \vdash \Delta}{\Gamma \mid \bar{\mu}[S] : \mathbf{0} \vdash \Delta} (\mathbf{0} \vdash_f)$$

(d) Logic

Figure 3: **IMALL**<sub>p</sub>: type system

While **LK**<sub>p</sub><sup>η</sup> was originally formulated without a term syntax, the same methodology was applied again using the Curien-Herbelin syntactic technology ( $\bar{\lambda}\mu\tilde{\mu}$ ) for sequent calculus (Curien and Herbelin, 2000; Herbelin, 2005) in Munch-Maccagnoni (2009). The calculus **IMALL**<sub>p</sub> is therefore essentially a linear and intuitionistic version of **LK**<sub>p</sub><sup>η</sup> developed with the Curien-Herbelin syntactic technology. In order to understand **IMALL**<sub>p</sub>, it is therefore crucial to first understand DJS's approach to the proof theory of **LK**.

The main idea behind DJS's noetherian cut-elimination is how they approach critical pairs encountered during the cut elimination of **LK** (such as the so-called “Lafont” critical pair involving

|                                                                                                                                                     |                                                                                                                                                                                                                                                                                       |
|-----------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| $\frac{\Gamma \vdash t : A \mid \Gamma' \mid e : B \vdash \Delta}{\Gamma, \Gamma' \mid t \cdot e : A \multimap B \vdash \Delta} (\multimap \vdash)$ | $t_{\varepsilon_1} \cdot e_{\varepsilon_2} \stackrel{\text{def}}{=} \bar{\mu}x^- . \langle \mu\alpha^{\varepsilon_2} . \langle t \parallel \bar{\mu}y^{\varepsilon_1} . \langle x \parallel y \cdot \alpha \rangle^- \rangle^{\varepsilon_1} \parallel e \rangle^{\varepsilon_2}$     |
| $\frac{\Gamma \vdash t : A \mid \Gamma' \vdash u : B \mid}{\Gamma, \Gamma' \vdash t \otimes u : A \otimes B \mid} (\vdash \otimes)$                 | $t_{\varepsilon_1} \otimes u_{\varepsilon_2} \stackrel{\text{def}}{=} \mu\alpha^+ . \langle t \parallel \bar{\mu}x^{\varepsilon_1} . \langle u \parallel \bar{\mu}y^{\varepsilon_2} . \langle x \otimes y \parallel \alpha \rangle^+ \rangle^{\varepsilon_2} \rangle^{\varepsilon_1}$ |
| $\frac{\Gamma \vdash t : A_i \mid}{\Gamma \vdash \iota_i(t) : A_1 \oplus A_2 \mid} (\vdash \oplus_i)$                                               | $\iota_i(t_\varepsilon) \stackrel{\text{def}}{=} \mu\alpha^+ . \langle t \parallel \bar{\mu}x^\varepsilon . \langle \iota_i(x) \parallel \alpha \rangle^+ \rangle^\varepsilon$                                                                                                        |
| $\frac{\Gamma \mid e : A_i \vdash \Delta}{\Gamma \mid \pi_i \cdot e : A_1 \& A_2 \vdash \Delta} (\&_i \vdash)$                                      | $(\pi_i \cdot e_\varepsilon) \stackrel{\text{def}}{=} \bar{\mu}x^- . \langle \mu\alpha^\varepsilon . \langle x \parallel \pi_i \cdot \alpha \rangle^- \parallel e \rangle$                                                                                                            |
| $\frac{}{\Gamma \mid \bar{\mu}[\text{dom}(\Gamma, \Delta)] : \mathbf{0} \vdash \Delta} (\mathbf{0} \vdash)$                                         | $\bar{\mu}[\{x_1, \dots, x_n, \alpha\}] \stackrel{\text{def}}{=} \bar{\mu}[x_1 \dots x_n \cdot \alpha]$                                                                                                                                                                               |
| $\frac{}{\Gamma \vdash \mu < \text{dom } \Gamma > : \top \mid} (\vdash \top)$                                                                       | $\mu < \{x_1, \dots, x_n\} > \stackrel{\text{def}}{=} \mu < x_1 \otimes \dots \otimes x_n >$                                                                                                                                                                                          |

Figure 4: Some derivable rules in **IMALL<sub>p</sub>**.

a cut between two formulae introduced by weakening). [DJS 1997](#) resolves these critical pairs in a way which is persistent (in fact determined by a typing annotation<sup>3</sup>), in the sense that it is fixed during cut-elimination, as a way of avoiding looping situations. These directions are and either towards the “head” (−) or towards the “tail” (+), in a terminology inspired by, and related to, that of “head reduction” in the  $\lambda$  calculus.

In the term syntax, annotations (−) or (+) correspond to opposite ways of determining priority in a cut:

$$\begin{aligned} \langle \mu\alpha^- . c \parallel \bar{\mu}x^- . c' \rangle^- &\triangleright_R c'[\mu\alpha^- . c/x] \\ \langle \mu\alpha^+ . c \parallel \bar{\mu}x^+ . c' \rangle^+ &\triangleright_R c[\bar{\mu}x^+ . c'/\alpha] \end{aligned}$$

Starting from this idea, it is possible to understand **LK<sub>p</sub><sup>η</sup>** as the system whose design follows the principles of *polarisation* and *focalisation*. To understand these design principles and where they come from, assume that we take as primitive the notions of 1) local reduction of principal cuts, and 2)  $\eta$ -expansion. Then, we ask that expansions “play well” with reductions, and derive consequences of this choice.

For instance in **IMALL**, one expects that a principal cut for  $\oplus$  on the left-hand side below reduces into, or at least has the same reduction behaviour, as the cut on the right-hand side below.

$$\frac{\frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B} \quad \frac{\Gamma', A \vdash \Delta \quad \Gamma', B \vdash \Delta}{\Gamma', A \oplus B \vdash \Delta}}{\Gamma', \Gamma \vdash \Delta} \quad \frac{\Gamma \vdash A \quad \Gamma', A \vdash \Delta}{\Gamma', \Gamma \vdash \Delta}$$

$$\langle \iota_1(t) \parallel \bar{\mu}[x.c \mid y.c'] \rangle \quad \langle t \parallel \bar{\mu}x.c \rangle$$

Similarly, and as importantly in the point of view of proof-relevant systems such as **LK<sub>p</sub><sup>η</sup>** and **IMALL<sub>p</sub>**, an arbitrary derivation for a sequent such that the one on the left-hand side below

<sup>3</sup>The first and most general system introduced in [DJS 1997](#), **LK<sup>tq</sup>**, has explicit and arbitrary annotations ( $t$ , or  $-$ ) and ( $q$ , or  $+$ ) on formulae. One can see this as being equivalent to a presentation with some default direction determined for each connective, together with “polarity-shifting” unary connectives imposing a (−) or (+) direction.

must have the same reduction behaviour (or meaning, in some sense), as its  $\eta$ -expanded version on the right-hand side below.

$$\begin{array}{c}
\Gamma, A \oplus B \vdash \Delta \quad \frac{\frac{A \vdash A}{A \vdash A \oplus B} \quad \Gamma, A \oplus B \vdash \Delta}{\Gamma, A \vdash \Delta} \quad \frac{\frac{B \vdash B}{B \vdash A \oplus B} \quad \Gamma, A \oplus B \vdash \Delta}{\Gamma, B \vdash \Delta} \\
\hline
\Gamma, A \oplus B \vdash \Delta
\end{array}$$

$$e \quad \bar{\mu}[x.\langle \iota_1(x) \parallel e \rangle \mid y.\langle \iota_2(y) \parallel e \rangle]$$

This assumption has two main consequences on the choices made during cut-elimination, indeed polarisation and focalisation:

- **Polarisation.** We would like that an expansion leaves the reduction order unchanged. Therefore when an expansion is available and seems to force a particular reduction, as in the following example of a cut of type  $\oplus$ :

$$\begin{aligned}
\langle \mu\alpha^\varepsilon.c \parallel \bar{\mu}x^\varepsilon.c' \rangle^\varepsilon &\rightarrow_E \langle \mu\alpha^\varepsilon.c \parallel \bar{\mu}[x.\langle \iota_1(x) \parallel \bar{\mu}x^\varepsilon.c' \rangle \mid y.\langle \iota_2(y) \parallel \bar{\mu}x^\varepsilon.c' \rangle] \rangle^\varepsilon \\
&\triangleright_R c[\bar{\mu}[x.\langle \iota_1(x) \parallel \bar{\mu}x^\varepsilon.c' \rangle \mid y.\langle \iota_2(y) \parallel \bar{\mu}x^\varepsilon.c' \rangle] / \alpha] \\
&\leftarrow_E^* c[\bar{\mu}x^\varepsilon.c' / \alpha],
\end{aligned}$$

then this must be the reduction that was determined all along by the annotation:

$$\langle \mu\alpha^\varepsilon.c \parallel \bar{\mu}x^\varepsilon.c' \rangle^\varepsilon \triangleright_R c[\bar{\mu}x^\varepsilon.c' / \alpha] \quad i.e. \ \varepsilon = +$$

Given that the connectives  $\mathbf{1}, \otimes, \oplus, \mathbf{0}$  have their expansions on the right-hand side of contexts as above, and the connectives  $\multimap, \&, \top$  have their expansion on the left-hand side of expressions, the former are  $(+)$  and the latter are  $(-)$ .

- **Focalisation.** An expansion can reveal reductions as in the following example, again of a cut of type  $\oplus$ :

$$\begin{aligned}
\langle \iota_1(t_+) \parallel \bar{\mu}x^+.c \rangle^+ &\rightarrow_{RE}^* \langle \iota_1(t_+) \parallel \bar{\mu}[y.c[\iota_1(y)/x] \mid z.c[\iota_2(z)/x]] \rangle^+ \\
&\rightarrow_R^* \langle t_+ \parallel \bar{\mu}y^+.c[\iota_1(y)/x] \rangle^+
\end{aligned}$$

where the reduction in last line is desired because it corresponds to a local reduction of a principal cut for  $\oplus$ . This is an evidence that  $\iota_1(t_+)$  contains a hidden cut, which we require to reveal itself without the intervention of an expansion:

$$\langle \iota_1(t_+) \parallel \bar{\mu}x^+.c \rangle^+ \rightarrow_R^* \langle t_+ \parallel \bar{\mu}y^+.c[\iota_1(y)/x] \rangle^+$$

In **IMALL**<sub>p</sub>, this is obtained by restricting the primitive form of constructors to values and co-values (e.g.  $\iota_1(V)$ ), which forces unrestricted forms (such as  $\iota_1(t_+)$ ) to be derived as in fig. 4, using a cut:

$$\iota_1(t_+) = \mu\alpha^+.\langle t \parallel \bar{\mu}x^+.\langle \iota_i(x) \parallel \alpha \rangle^+ \rangle^+.$$

In terms of focused (*i.e.* normal) proofs, focalisation is responsible for the atomicity of synchronous phases.

**IMALL**<sub>p</sub> is obtained by starting from the Curien-Herbelin syntactic toolkit and applying these ideas: in this sense it is a proof-relevant **IMALL** whose theory of noetherian proof reduction and semantics-preserving proof equivalence derives from the principles of polarisation and focalisation we just described.

The strength of this Curry-Howard take on Andreoli's focusing will become manifest in Section 2.2, as we will see that the same design principles still work for the modalities  $!$  and  $\Diamond$ , although the result is different: in particular an interpretation of  $!$  and  $\Diamond$  as relative monads and comonads will appear.

## 1.4. The syntactic model of linear and thunkable terms

The syntax in fact defines a syntactic model of linear CBPV, via a construction for which we need the following definition first.

**Definition 2.** A morphism  $f$  in a non-associative category is **thunkable** if any composition chain  $h \circ g \circ f$  associates. A morphism  $h$  in a non-associative category is **linear** if any composition chain  $h \circ g \circ f$  associates.

Thunkable and linear morphisms determine (associative) sub-categories of the non-associative category. We call  $\mathcal{P}_t$  the subcategory of positive objects and thunkable maps of  $\mathbf{IMALL}_p$  seen as a deductive system with  $\simeq_{\text{RE}}$  as equality of proofs, and  $\mathcal{N}_l$  its subcategory of negative objects and linear maps. An advantage of considering thunkable and linear maps is that we do not need to define so-called “complex” values and stacks as in Levy (2004).

**Proposition 3.** *The  $\mathbf{IMALL}_p$  calculus defines a syntactic linear CBPV model*

$$\begin{array}{ccc} & \xrightarrow{1 \multimap -} & \\ \mathcal{P}_t & \perp & \mathcal{N}_l \\ & \xleftarrow{1 \otimes -} & \end{array}$$

with the symmetric monoidal and distributive structure on  $\mathcal{P}_t$  given by the connectives  $\otimes, \mathbf{1}$  and  $\oplus, \mathbf{0}$ , and the cartesian closure relative to  $\mathcal{P}_t$  on  $\mathcal{N}_l$  given by the connectives  $\&, \top$  and  $\multimap$ .

We therefore have two ways of organising  $\mathbf{IMALL}_p$  categorically with cut as composition and  $\simeq_{\text{RE}}$  as equality of morphisms: 1) as deductive systems, i.e. non-associative categories, and 2) as linear CBPV model. The two interpretations turn out to be equivalent, in the sense that the non-associative category arising from the adjunctions between positive thunkable and negative linear terms is equivalent to the original non-associative category (Munch-Maccagnoni, 2014).<sup>4</sup>

## 1.5. Adjunctions in non-associative categories

The notion that reductions and expansions have to “play well” together is traditionally characterised in categorical semantics with the presence of adjunctions, in which connectives are left or right adjoints. One recent realisation is that this characterisation already makes sense in bare deductive systems: non-associative categories. Let us recall the notion of adjunction between bare functors over non-associative categories introduced in MMMM 2026.

**Definition 4.** A **bare functor**  $F : \mathcal{D} \rightarrow \mathcal{E}$  between non-associative categories  $\mathcal{D}$  and  $\mathcal{E}$  is a morphism of the reflexive graphs  $\mathcal{D}$  and  $\mathcal{E}$ , in other words a bare functor  $F : \mathcal{D} \rightarrow \mathcal{E}$  is a functor that does not necessarily preserve composition. A **proper** functor is one that does (notation  $\mathcal{D} \rightarrow \mathcal{E}$ ).

**Definition 5.** An **adjunction** over non-associative categories  $\mathcal{D}$  and  $\mathcal{E}$

$$\begin{array}{ccc} & \xrightarrow{F} & \\ \mathcal{D} & \perp & \mathcal{E} \\ & \xleftarrow{G} & \end{array}$$

<sup>4</sup>The syntactic result suggests in fact that the structure theorem from Führmann (2000) and Munch-Maccagnoni (2014) extends to linear CBPV, which is a result that we expect.

|                                                                                                                     |                                                                                                                                                                             |
|---------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| positive $P, Q, A_+ ::= \dots \mid !A$                                                                              |                                                                                                                                                                             |
| negative $N, M, A_- ::= \dots \mid \Diamond A$                                                                      | • $!\Gamma$ stands for a typing context $!A_1, \dots, !A_n$ .                                                                                                               |
| (a) Types                                                                                                           | (b) Judgements                                                                                                                                                              |
| $\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}^{(w)} \quad \frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}^{(c)}$ | $\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}^{(!\vdash)} \quad \frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}^{(!\vdash)}$                                                   |
| (c) Structure                                                                                                       | $\frac{\Gamma \vdash A}{\Gamma \vdash \Diamond A}^{(\vdash \Diamond)} \quad \frac{!\Gamma, A \vdash \Diamond B}{!\Gamma, \Diamond A \vdash \Diamond B}^{(\Diamond \vdash)}$ |
|                                                                                                                     | (d) Logic                                                                                                                                                                   |

Figure 5:  $\mathbf{ILL}^\Diamond = \mathbf{IMALL} +$  the rules above

consists of two bare functors  $F : \mathcal{D} \rightarrow \mathcal{E}$  and  $G : \mathcal{E} \rightarrow \mathcal{D}$  and a family of bijections:

$$\varphi_{A,B} : \mathcal{E}(FA, B) \cong \mathcal{D}(A, GB)$$

natural separately in  $A \in \mathcal{D}$  and  $B \in \mathcal{E}$ , that is to say, for any composition chain  $g \circ h \circ Ff$  in  $\mathcal{E}$  (not necessarily associative), where  $h$  is an oblique morphism, one has  $\varphi(g \circ h) = Gg \circ \varphi(h)$  and  $\varphi(h \circ Ff) = \varphi(h) \circ f$  for any oblique morphism  $f$ .

Note that an adjunction over associative categories is an adjunction in the usual sense, since the bare functors are necessarily proper in that case.

**Definition 6.** For any pair of non-associative categories  $\mathcal{D}, \mathcal{E}$ , their **direct product**  $\mathcal{D} \times \mathcal{E}$  is the non-associative category with objects  $|\mathcal{D}| \times |\mathcal{E}|$  and morphisms  $(\mathcal{D} \times \mathcal{E})((A, A'), (B, B')) = \mathcal{D}(A, B) \times \mathcal{E}(A', B')$  defined in the obvious way. For  $\mathcal{D}$  a non-associative category, we define the **diagonal functor**  $\Delta : \mathcal{D} \rightarrow \mathcal{D} \times \mathcal{D}$  with  $\Delta A = (A, A)$  and  $\Delta f = (f, f)$ .

Note that  $\Delta$  preserves both identities and composition; it is in fact a proper functor.

**Proposition 7.** In  $\mathbf{IMALL}_p$  considered as a non-associative category  $\mathcal{D}$ , the diagonal functor  $\Delta$  has a left adjoint  $\hat{\oplus} : \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}$  with  $A \hat{\oplus} B = A \oplus B$  on objects, satisfying a family of isomorphisms:

$$(A \oplus B \vdash C) \cong (A \vdash C) \times (B \vdash C)$$

natural in  $(A, B) \in (\mathcal{D} \times \mathcal{D})^{\text{op}}$  and in  $C \in \mathcal{D}$  separately.

The functor  $\Delta$  also has a right adjoint  $\hat{\&} : \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}$  with  $A \hat{\&} B = A \& B$  on objects, satisfying a family of isomorphisms:

$$(A \vdash B \& C) \cong (A \vdash B) \times (A \vdash C)$$

natural in  $A \in \mathcal{D}^{\text{op}}$  and in  $(B, C) \in \mathcal{D} \times \mathcal{D}$  separately.

More details are given in the case of sums in Section C (the case of the product  $\&$  is symmetric). The relevant structure of  $\otimes$  and  $\multimap$  is described in [MMMM 2026](#); we omit it in this paper for simplicity.

## 2. $\mathbf{ILL}^\diamond$ : Resource and effect modalities

In this section, we extend the previous development, starting from a sequent calculus for multiplicative additive intuitionistic linear logic  $\mathbf{IMALL}$  now with a resource or exponential modality noted  $A \mapsto !A$ ; in other words the full propositional intuitionistic linear logic  $\mathbf{ILL}$ , what we write

$$\mathbf{ILL} = \mathbf{IMALL} + !$$

We also consider the combination of  $\mathbf{ILL}$  with an effect modality noted  $A \mapsto \diamond A$  required to be strong with respect to the resource modality, in the sense that it comes with the axiom schema

$$!A \otimes \diamond B \vdash \diamond(!A \otimes B)$$

We obtain in that way the intuitionistic modal logic

$$\mathbf{ILL}^\diamond = \mathbf{ILL} + \diamond = \mathbf{IMALL} + ! + \diamond$$

whose sequent calculus is presented in Figure 5. Note that  $\mathbf{ILL}^\diamond$  may be seen as a linear variant of the constructive modal logic  $\mathbf{CS4}$  (Alechina, Mendler et al., 2001) where we find convenient (and in the tradition of linear logic) to write  $A \mapsto !A$  for the necessity modality  $A \mapsto \Box A$ . We replay the developments from the previous section.

### 2.1. Linear CBPV models with resource and effect modalities

As in Proposition 1 for  $\mathbf{IMALL}$ , there is an interpretation of linear CBPV models as deductive systems that validate the rules of Figure 5. Let us first recall that a linear CBPV model with a resource modality (CFMM 2016) is given by a linear CBPV adjunction (1) together with a linear-non-linear adjunction

$$\begin{array}{ccc} \mathcal{M} & \xrightleftharpoons[\text{Mult}]{\text{Lin}} & \mathcal{V} \end{array} \quad \begin{array}{ccc} & \xrightleftharpoons[\downarrow]{\uparrow} & \mathcal{S} \end{array} \quad (2)$$

that is, a symmetric monoidal adjunction as above where the monoidal structure on  $\mathcal{M}$  is cartesian, leading to:

**Proposition 8** (CFMM 2016). *The deductive system  $\mathbf{dupl}_{\uparrow, \downarrow}$  associated to a linear CBPV adjunction with resource modality (2) satisfies the rules of  $\mathbf{ILL}$ , that is to say the rules of Figure 5 without the modality  $\diamond$ .*

Interpreting all the rules from fig. 5 in an analogous of Proposition 8 for  $\mathbf{ILL}^\diamond$  is possible and requires an additional structure in the form of an effect modality: something like a monad  $T = G \circ F$  on  $\mathcal{S}$  which is strong with respect to an action of  $\mathcal{M}^{\text{op}}$  on  $\mathcal{S}$  (Melliès, 2012b) leading to a series of “strong” adjunctions

$$\begin{array}{ccccc} \mathcal{M} & \xrightleftharpoons[\text{Mult}]{\text{Lin}} & \mathcal{V} & \xrightleftharpoons[\downarrow]{\uparrow} & \mathcal{S} & \xrightleftharpoons[G]{F} & \mathcal{C} \end{array} \quad (3)$$

In order to make it no more complicated than necessary for our purposes, here with  $F \dashv G$  being strong we mean that  $T$  is provided with a natural family of morphisms in  $\mathcal{V}$ ,

$$lstr_{X,P} : Lin X \otimes \downarrow T \uparrow P \rightarrow \downarrow T \uparrow (Lin X \otimes P),$$

making four usual diagrams commute—corresponding to the notion of *strength with respect to*  $Lin : \mathcal{M} \rightarrow \mathcal{V}$  (Blute, Cockett and Seely, 1996; Hasegawa, 2002) for the monad  $\downarrow \circ T \circ \uparrow$  on  $\mathcal{V}$  obtained by composing adjunctions. This notion might seem somewhat restrictive, and it is going to be all the more interesting that this is enough for our purposes.

**Example 9.**

- Any (non-polarised) model of intuitionistic linear logic, that is to say a situation (2) in which the symmetric monoidal category  $\mathcal{V}$  is closed,  $\mathcal{S} = \mathcal{V}$ , and  $\uparrow = \downarrow = \text{Id}_{\mathcal{V}}$ , has a situation (3) in which  $T$  is the exception monad  $\mathcal{E} = - \oplus E$  (for any  $E \in \mathcal{V}$ ), which is strong with respect to  $Lin$ , in the sense that the linear tautologies

$$!\Gamma \otimes (A \oplus E) \vdash (!\Gamma \otimes A) \oplus E$$

underlie a natural family of morphisms

$$Lin X \otimes \mathcal{E} A \rightarrow \mathcal{E}(Lin X \otimes A)$$

satisfying the four coherence laws. Diagrammatically:

$$\begin{array}{ccccc} \mathcal{M} & \xrightarrow{Lin} & \mathcal{V} & \xrightarrow{F^{\mathcal{E}}} & \mathcal{V}^{\mathcal{E}} \\ & \perp & & \perp & \\ & \xleftarrow{Mult} & & \xleftarrow{G^{\mathcal{E}}} & \end{array}$$

- Any model of linear logic, that is to say (2) in which  $\mathcal{V}$  is  $*$ -autonomous,  $\mathcal{S} = \mathcal{V}$ , and  $\uparrow = \downarrow = \text{Id}$ , has a situation (3) in which  $T = ?$  is the dual of  $!$ ,  $\mathcal{C} = \mathcal{M}^{\text{op}}$ ,  $F = Mult \circ \neg$  and  $G = \neg \circ Lin$ , and in which the linear tautologies

$$!\Gamma \otimes ?A \vdash ?(!\Gamma \otimes A)$$

underlie again a strength of  $?$  in the previous sense.

$$\begin{array}{ccccc} \mathcal{M} & \xrightarrow{Lin} & \mathcal{V} & \xrightarrow{Mult \circ \neg} & \mathcal{M}^{\text{op}} \\ & \perp & & \perp & \\ & \xleftarrow{Mult} & & \xleftarrow{\neg \circ Lin} & \end{array}$$

- By not assuming  $\mathcal{V}$  to be  $*$ -autonomous but merely a dialogue category (a symmetric monoidal category having a negation functor  $\neg : \mathcal{V} \rightarrow \mathcal{V}^{\text{op}}$ , in the sense of having an exponentiable object), we arrive at a linear CBPV model with a resource modality (2) with  $\mathcal{S} = \mathcal{V}^{\text{op}}$  and with the adjunction of negation with itself. Such models give again rise to  $\mathbf{ILL}_p^{\diamond}$  models (3) with  $\mathcal{C} = \mathcal{M}^{\text{op}}$ ,  $F = Mult$  and  $G = Lin$ .

$$\begin{array}{ccccccc} \mathcal{M} & \xrightarrow{Lin} & \mathcal{V} & \xrightarrow{\neg} & \mathcal{V}^{\text{op}} & \xrightarrow{Mult} & \mathcal{M}^{\text{op}} \\ & \perp & & \perp & & \perp & \\ & \xleftarrow{Mult} & & \xleftarrow{\neg} & & \xleftarrow{Lin} & \end{array} \quad (4)$$

This situation has been used to explain translations of (polarised) classical logic into linear logic and to describe game models of linear logic in Melliès and Tabareau (2010).

**Proposition 10.** *The deductive system  $\mathbf{dupl}_{\uparrow, \downarrow}$  associated to a linear CBPV adjunction with resource and effect modalities (3) satisfies the rules of  $\mathbf{ILL}^\diamond$  of Figure 5.*

Up until now, we did not detail the interpretation of formulæ of  $\mathbf{IMALL}$  as objects of linear CBPV models since it was rather straightforward. However, the interpretation is no longer straightforward for  $!$  and  $\diamond$ . Indeed, the valid interpretation of the rules for  $!$  and  $\diamond$  require an extra polarity shift:

$$\begin{aligned} !A &= \text{Lin} \circ \text{Mult} \circ \downarrow(A^-) \\ \diamond A &= T \circ \uparrow(A^+) \end{aligned} \tag{5}$$

This phenomenon corresponds to the fact that starting from an oblique morphism

$$\mathcal{O}(\text{Lin } X_1 \otimes \cdots \otimes \text{Lin } X_n, \uparrow P)$$

one cannot in general obtain an oblique morphism

$$\mathcal{O}(\text{Lin } X_1 \otimes \cdots \otimes \text{Lin } X_n, \uparrow \text{Lin Mult } P),$$

merely one in

$$\mathcal{O}(\text{Lin } X_1 \otimes \cdots \otimes \text{Lin } X_n, \uparrow \text{Lin Mult } \downarrow P). \tag{6}$$

This phenomenon was described semantically in Curien, Fiore and Munch-Maccagnoni (CFMM 2016) but was observed syntactically as early as Andreoli (1992) with the fact that the introduction of  $!$  terminates a synchronous phase and that the introduction of  $?$  (or  $!$  on the left) terminates an asynchronous phase in focusing proof search. The monad  $T$  undergoes a dual phenomenon. But these additional shifts in (5) have consequences on the completeness of the interpretation, as we will see.

## 2.2. The calculus $\mathbf{ILL}_p^\diamond$ : A polarised syntax for $!$ and $\diamond$ modalities

The calculus  $\mathbf{ILL}_p^\diamond$  extending  $\mathbf{IMALL}_p$  is defined in Figure 6. It is  $\mathbf{IMALL}_p$  with modalities  $!$  and  $\diamond$  where we use a syntax introduced for linear logic exponentials  $!, ?$  in Munch-Maccagnoni (2009). One of the goals of this paper is to explain this somewhat unintuitive, but mathematically justified syntax.

The calculus  $\mathbf{ILL}_p^\diamond$  enjoys the same good properties we mentioned for  $\mathbf{IMALL}_p$ : its type system expresses the logic  $\mathbf{ILL}^\diamond$ , and its derivations can be soundly interpreted in any linear CBPV model with a resource and an effect modalities in the non-relative sense of (3). The reduction relation  $\rightarrow_R$  defines again a noetherian and semantics-preserving cut-elimination strategy for  $\mathbf{ILL}^\diamond$ . The  $\rightarrow_R$ -normal and  $\rightarrow_E$ -long terms are again in correspondence with standard focused normal forms, from which the completeness of focusing proof search follows (see Appendix E).

## 2.3. An explanation of $\mathbf{ILL}_p^\diamond$ from a proof-theoretic perspective

Now we turn to the design constraints related to polarisation and focalisation that gave rise to this syntax. As those are concerned with playing well with  $\eta$  rules, the first observation is that the  $\eta$ -rules for  $!$  and  $\diamond$  are very different to other connectives. For instance, in terms of mere provability, none of their left- and right-introduction rules are invertible. There is a weaker property of “semi-invertibility” observed in Laurent (2002): provided that the context  $\Gamma$  is of the right form, the modality  $!$  is invertible on the right in terms of mere provability.

For a notion of  $\eta$  rule which is justified semantically (not merely in terms of provability), we have to adopt a notion of invertibility (or  $\eta$ -expansion) for  $!$  and  $\diamond$  which is weaker still. It relies

$$\begin{array}{lcl}
\text{Values: } V, W ::= & \dots & \mid \mu! \alpha. c \mid \diamond V \\
\text{Stacks: } S ::= & \dots & \mid !S \mid \bar{\mu} \diamond x. c
\end{array}$$

(a) Terms

$$\begin{array}{ll}
(R!) : \langle \mu! \alpha. c \parallel !S \rangle^+ \triangleright_R c[S/\alpha] & (E!) : V \triangleright_E \mu! \alpha. \langle V \parallel !\alpha \rangle^+ \\
(R\diamond) : \langle \diamond V \parallel \bar{\mu} \diamond x. c \rangle^- \triangleright_R c[V/x] & (E\diamond) : S \triangleright_E \bar{\mu} \diamond x. \langle \diamond x \parallel S \rangle^+
\end{array}$$

(b) Reduction rules (c) Extensionality rules

- $! \Gamma$  stands for a typing context  $x_1 : !A_1, \dots, x_n : !A_n$ .
  - $\diamond \Delta$  stands for a typing context  $\alpha : \diamond A$ .
  - $\Sigma^!(\Gamma; \Gamma')$  is the subset of  $\Sigma(\Gamma; \Gamma')$  of maps that are bijective on variables not of the form  $!A$ .
  - The structural rules for  $\mathbf{ILL}_p^\diamond$  are given by taking  $\Sigma^!(\Gamma; \Gamma')$  for the range of structural maps (renaming and exchange, as well as weakening and exchange for formulae of the form  $!A$ ).
- (d) Judgements (e) Structure

$$\begin{array}{ll}
\frac{c : (!\Gamma \vdash \alpha : A)}{!\Gamma \vdash \mu! \alpha. c : !A} (\vdash !) & \frac{\Gamma \mid S : A \vdash \Delta}{\Gamma \mid !S : !A \vdash \Delta} (! \vdash_f) \\
\frac{\Gamma \vdash V : A}{\Gamma \vdash \diamond V : \diamond A} (\vdash_f \diamond) & \frac{c : (!\Gamma \mid x : A \vdash \diamond \Delta)}{!\Gamma \mid \bar{\mu} \diamond x. c : \diamond A \vdash \diamond \Delta} (\diamond \vdash)
\end{array}$$

(f) Logic

$$\begin{array}{ll}
\frac{\Gamma \mid e : A \vdash \Delta}{\Gamma \mid !e : !A \vdash \Delta} (! \vdash) & !e_\varepsilon \stackrel{\text{def}}{=} \bar{\mu} x^+. \langle \mu \alpha^\varepsilon. \langle x \parallel !\alpha \rangle \parallel e \rangle \\
\frac{\Gamma \vdash t : A}{\Gamma \vdash \diamond t : \diamond A} (\vdash \diamond) & \diamond t_\varepsilon \stackrel{\text{def}}{=} \mu \alpha^-. \langle t \parallel \bar{\mu} x^\varepsilon. \langle \diamond \alpha \parallel x \rangle \rangle
\end{array}$$

(g) Additional derived rules in  $\mathbf{ILL}_p^\diamond$ .

Figure 6:  $\mathbf{ILL}_p^\diamond = \mathbf{IMALL}_p + \text{above}$

on the following principle: a well-typed value of type  $!A$  must come from either a promotion or a variable (possibly with some structural rules applied); intuitively it corresponds to a “box” in terms of proof nets for linear logic. In particular the context of this value is of the form  $!\Gamma$ . The weak invertibility principle for  $!$  is then reflected in the syntax with a promotion constructor in the shape of pattern matching together with a dereliction constructor in the shape of a pattern:

$$!\Gamma \vdash \mu! \alpha. c \quad \Gamma \mid !e : !A \vdash \Delta$$

in such a way that the corresponding expansion correctly reflects a (weak) invertibility of  $!$  on the right:

$$V \triangleright_E \mu! \alpha. \langle V \parallel !\alpha \rangle^+ \tag{7}$$

and that it is well-typed. Dually, a well-typed co-value of type  $\diamond A$  must come from either an extension or a covariable, such that its context is of the form  $!\Gamma, \diamond \Delta$ , in such a way that the

expansion

$$S \triangleright_E \bar{\mu} \diamond x. \langle \diamond x \parallel S \rangle^+$$

is well-typed.<sup>5</sup> Starting from these desired  $\eta$ -expansions, polarisation and focalisation again help us design the rest of the calculus.

- **Polarisation.** We first notice that the weak invertibility property is not sufficient to make  $!$  a negative connective (and dually to make  $\diamond$  to be a positive connective) since they apply conditionally, and thus cannot be used to force a reduction. Other constraints require  $!$  to be positive (and dually  $\diamond$  negative): namely to ensure that values of type  $!A$  and covalues of type  $\diamond A$  are “boxes” – (co)algebra morphisms – i.e. promotions or variables for  $!$ , extension or covariable for  $\diamond$ . To illustrate this requirement, consider in a generic cut

$$\langle t \parallel \bar{\mu} x^\varepsilon. c \rangle^\varepsilon$$

where  $t$  is an expression of type  $!A$ , and where  $x$  might occur several or zero times in  $c'$ . We do not know that  $t$  is allowed to be duplicated or erased – i.e. is a box, or a  $!$ -coalgebra morphism. In general this might even be ill-typed since the context of  $t$  is not necessarily of the form  $!\Gamma$ . The solution is indeed to assign the polarity  $\varepsilon = +$  to the modality  $!$ , enforcing a call-by-value reduction:

$$\begin{aligned} \text{case } t = \mu \alpha^+. c' : & \quad \langle \mu \alpha^+. c' \parallel \bar{\mu} x^+. c \rangle^+ \triangleright_R c'[\bar{\mu} x^+. c / \alpha] \\ \text{case } t = V \neq \mu \alpha^+. c' : & \quad \langle V \parallel \bar{\mu} x^+. c \rangle^+ \triangleright_R c[V / \alpha] \end{aligned}$$

This has the crucial consequence that duplication and erasure (and expansion (7)) can only happen for  $V$  of the form  $V = x$  or  $V = \mu! \alpha. c$ . Dually, assigning the polarity  $\varepsilon = -$  to the modality  $\diamond$  enforces a call-by-name reduction meaning that a covalue  $S$  of type  $\diamond A$  is necessarily some  $S = \bar{\mu} \diamond x. c$  or some  $S = \alpha$ .

- **Focalisation.** The introduction rule for  $!$  on the right (respectively  $\diamond$  on the left) does not admit a restriction to values (resp. stacks). Indeed, it is not possible to derive this rule from the same rule restricted to a value (resp. a stack) using cuts, unlike for the positive connectives  $\oplus$  and  $\otimes$  on the right, like  $\iota_1 t$  derives from  $\iota_1 V$  (resp. for the negative connectives  $\multimap$  and  $\&$  on the left). These rules have a suspending effect. This one way we can interpret our observation on oblique morphisms (6) from the introduction, forcing extra shifts in the interpretation into the models. However, weak focalisation properties are available for  $!$  on the left (dereliction) and  $\diamond$  on the right (return), forced by the weak invertibility on the opposite side, as follows:

$$\begin{aligned} \langle V \parallel !e_- \rangle^+ &\rightarrow_E \langle \mu! \alpha. \langle V \parallel !\alpha \rangle^+ \parallel !e_- \rangle^+ \rightarrow_R^* \langle \mu \alpha^-. \langle V \parallel !\alpha \rangle^+ \parallel e_- \rangle^- \\ \langle \diamond t_+ \parallel S \rangle^- &\rightarrow_E \langle \diamond t_+ \parallel \bar{\mu} \diamond x. \langle \diamond x \parallel S \rangle^- \rangle^- \rightarrow_R^* \langle t_+ \parallel \bar{\mu} x^+. \langle \diamond x \parallel S \rangle^- \rangle^+ \end{aligned}$$

The weak focalisations implementing the derived equivalences above for dereliction and return are given in fig. 6g:

$$\begin{aligned} \langle V \parallel !e_- \rangle^+ &\rightarrow_R \langle \mu \alpha^-. \langle V \parallel !\alpha \rangle^+ \parallel e_- \rangle^- \\ \langle \diamond t_+ \parallel S \rangle^- &\rightarrow_R \langle t_+ \parallel \bar{\mu} x^+. \langle \diamond x \parallel S \rangle^- \rangle^+ \end{aligned}$$

<sup>5</sup>and whose interpretations into linear CBPV models are sound due to a stronger observation: the interpretation of expressions of type  $!A$  that are values are  $!$ -coalgebra morphisms whereas the interpretation of coexpressions of type  $\diamond A$  that are covalues are  $\diamond$ -algebra morphisms, although the initial motivations were syntactic (in terms of proof net boxes for  $!$  and  $?$  in linear logic, as explained).

These focalisation properties are weak in the sense that the following unrestricted focalisation-like rules are unsound in general:

$$\begin{aligned} \langle t_+ \parallel !e_- \rangle^+ &\rightarrow_{(\text{unsound})} \langle \mu\alpha^-. \langle t_+ \parallel !\alpha \rangle^+ \parallel e_- \rangle^- \\ \langle \diamond t_+ \parallel e_- \rangle^- &\rightarrow_{(\text{unsound})} \langle t_+ \parallel \bar{\mu}x^+. \langle \diamond x \parallel e_- \rangle^- \rangle^+. \end{aligned}$$

The calculus  $\mathbf{ILL}_p^\diamond$  amounts to  $\mathbf{IMALL}_p$  augmented with a resource modality  $!$  and an effect modality  $\diamond$  following the design constraints of polarisation and focalisation described above, guided again from the idea of valid reductions and expansions playing well together.

The calculus  $\mathbf{ILL}_p^\diamond$  and its specific syntax for  $!$  and  $\diamond$  are justified by a sound interpretation into linear call-by-push-value models with resource and effect modalities, and indeed serve to prove (a proof-relevant version of) the completeness of Andreoli's focusing proof search adapted to  $\mathbf{ILL}^\diamond$  (Appendix E).

We will now seek to recover the completeness property missing from CFMM 2016 by moving to relative (co)monads.

## 2.4. Linear call-by-push-value models with relative resource and effect modalities

By moving from  $\mathbf{IMALL}_p$  to  $\mathbf{ILL}_p^\diamond$  (and even merely the fragment  $\mathbf{ILL}_p$  without  $\diamond$ ), we lose completeness: it does not seem possible to recover a resource modality on the category of thunkables nor an effect modality on the category of linears. From the syntax, one can build in fact a syntactic linear CBPV model with a resource comonad relative to  $\downarrow$  and a monad relative to  $\uparrow$ , strong with respect to the comonad, as defined next. Monads (respectively comonads) relative to  $\uparrow$  (resp.  $\downarrow$ ) were previously used for modelling effect (resp. resource) modalities in Jiang, Xue and New (2025) (resp. Mellies (2025)).

**Definition 11** (Altenkirch, Chapman and Uustalu, 2015). A relative monad  $T$  on a functor  $L : \mathcal{A} \rightarrow \mathcal{B}$  is given by

- a mapping on objects  $T : |\mathcal{A}| \rightarrow |\mathcal{B}|$ ,
- for any  $A \in \mathcal{A}$ , a map  $\eta_A \in \mathcal{B}(LA, TA)$ ,
- for any  $A, B \in \mathcal{A}$ , and any  $f \in \mathcal{B}(LA, TB)$ , a map  $f^* \in \mathcal{B}(TA, TB)$ ,

such that two unital laws and one associativity law are satisfied:

$$f = f^* \circ \eta \qquad \eta_A^* = \text{id}_{TA} \qquad (f^* \circ g)^* = f^* \circ g^*$$

By duality:

**Definition 12.** A relative comonad  $S$  on a functor  $R : \mathcal{B} \rightarrow \mathcal{A}$  is given by:

- a mapping on objects  $S : |\mathcal{B}| \rightarrow |\mathcal{A}|$ ,
- for any  $B \in \mathcal{B}$ , a map  $\varepsilon_B \in \mathcal{A}(SB, B)$ ,
- for any  $A, B \in \mathcal{B}$ , and any  $f \in \mathcal{A}(SA, RB)$ , a map  ${}^*f \in \mathcal{A}(SA, SB)$ ,

such that two unital laws and one associativity law are satisfied:

$$f = \varepsilon \circ {}^*f \qquad {}^*\varepsilon_B = \text{id}_{SB} \qquad {}^*(f \circ {}^*g) = {}^*f \circ {}^*g$$

We will in fact use the characterisation of relative monads and comonads as relative adjunctions and coadjunctions from Altenkirch, Chapman and Uustalu (2015) and Arkor and McDermott (2024).

**Definition 13** (Relative adjunctions and coadjunctions on categories). Let  $\mathcal{A}$  and  $\mathcal{B}$  be two categories and  $L : \mathcal{A} \rightarrow \mathcal{B}$  be a functor. An  **$L$ -relative adjunction**  $F \dashv_L G$  consists of a category  $\mathcal{C}$ , two functors  $F : \mathcal{A} \rightarrow \mathcal{C}$  and  $G : \mathcal{C} \rightarrow \mathcal{B}$ , and a family of bijections:

$$\Phi_{A,C} : \mathcal{C}(FA, C) \cong \mathcal{B}(LA, GC).$$

natural in  $A$  and  $C$ . Diagrammatically:

$$\begin{array}{ccc} & \mathcal{C} & \\ F \nearrow & \dashv & \searrow G \\ \mathcal{A} & \xrightarrow{L} & \mathcal{B} \end{array} \quad (8)$$

We note  $\eta : L \rightarrow G \circ F$  the unit of the adjunction.

Dually, for a functor  $R : \mathcal{B} \rightarrow \mathcal{A}$ , an  **$R$ -relative coadjunction**  $F \dashv_R G$  consists of a category  $\mathcal{C}$ , two functors  $F : \mathcal{C} \rightarrow \mathcal{A}$  and  $G : \mathcal{B} \rightarrow \mathcal{C}$ , and a family of bijections:

$$\Psi_{C,B} : \mathcal{A}(FC, RB) \cong \mathcal{C}(C, GB)$$

natural in  $B$  and  $C$ . Diagrammatically:

$$\begin{array}{ccc} & \mathcal{C} & \\ F \nearrow & \dashv & \searrow G \\ \mathcal{A} & \xleftarrow{R} & \mathcal{B} \end{array} \quad (9)$$

We note  $\varepsilon : F \circ G \rightarrow R$  the counit of the coadjunction.

We can now revisit the interpretation from Section 2.1 in light of the notions of relative adjunction and coadjunction starting from the following simple fact about relative (co)adjunctions:

**Proposition 14.** *In (5) one has a  $\uparrow$ -relative coadjunction  $Lin \dashv_{\uparrow} Mult \circ \uparrow$ , and dually, adjoint resolutions  $F \dashv G$  of a comonad  $T$  on  $\mathcal{S}$  give rise to  $\downarrow$ -relative adjunctions  $F \circ \downarrow \dashv_{\downarrow} G$ .*

We observe that our interpretation of  $\mathbf{ILL}^\diamond$  into linear CBPV models (5) makes only use of this relative structure. Indeed, assume we are given on top of a linear CBPV adjunction (1)

- a  $\uparrow$ -relative adjunction

$$Lin \dashv_{\uparrow} Mult : \mathcal{S} \rightarrow \mathcal{M}$$

where  $Lin : \mathcal{M} \rightarrow \mathcal{V}$  is strong monoidal, defining a relative symmetric monoidal comonad  $! = Lin \circ Mult$ , and

- a  $\downarrow$ -relative adjunction

$$F \dashv_{\downarrow} G : \mathcal{V} \rightarrow \mathcal{C}$$

defining a relative monad  $\diamond = G \circ F$ , together with a strength with respect to  $Lin$ , in the form of a natural family of morphisms

$$str_{X,P} : Lin X \otimes \downarrow \diamond P \rightarrow \downarrow \diamond (Lin X \otimes P) \quad (10)$$

making the four usual diagrams commute.

Diagrammatically:

$$\begin{array}{ccccc}
 & & F & & \\
 & \nearrow & & \searrow & \\
 \mathcal{M} & \xrightarrow{Lin} & \mathcal{V} & \xrightarrow{\uparrow} & \mathcal{S} & \xrightarrow{F} & \mathcal{C} \\
 & \searrow & & \nearrow & & \nwarrow & \\
 & & \perp & & \perp & & \perp \\
 & \nwarrow & & \swarrow & \nwarrow & & \nwarrow \\
 & & \mathcal{V} & \xrightarrow{\downarrow} & \mathcal{S} & \xrightarrow{G} & \mathcal{C} \\
 & \nearrow & & \searrow & & \nearrow & \\
 & & Mult & & & & 
 \end{array} \tag{11}$$

**Definition 15.** We call the data (11) given by an  $\mathbf{IMALL}_p$  (linear CBPV) model from [CFMM 2016](#) together with a  $\uparrow$ -relative symmetric monoidal adjunction and a  $\downarrow$ -relative adjunction strong in the sense of (10) an  $\mathbf{ILL}_p^\diamond$  model.

Then one has:

**Proposition 16.** The syntactic linear CBPV model  $\mathcal{P}_t \rightleftarrows \mathcal{N}_l$  of thunkable and linear terms of  $\mathbf{ILL}_p^\diamond$  has a relative comonad  $! : \mathcal{N}_l \rightarrow \mathcal{P}_t$  on  $\Downarrow$ , and a relative monad  $\diamond : \mathcal{P}_t \rightarrow \mathcal{N}_l$  on  $\Uparrow$ , given by the corresponding rules of  $!$  and  $\diamond$ .

The idea is that thunkable terms  $!N \vdash \Downarrow M$  in the syntactic model are in correspondence to generic terms of type  $!N \vdash M$ , in such a way that the co-unit matches the dereliction restricted to covalues,

$$| \alpha : !N \vdash \alpha : N$$

and the coextension matches promotion, which indeed builds a thunkable morphism:

$$\frac{c : (!N \vdash \alpha : M)}{!N \vdash \mu! \alpha.c : !M} |$$

For more details consult [Appendix D](#). It is another sign and testimony of the fundamental nature of linear-non-linear relative coadjunctions introduced in the polarised setting of tensorial logic and functorial game semantics that they also provide a way to resolve a completeness issue appearing in the syntax and semantics of linear CBPV with resource modalities ([CFMM 2016](#)).

We now establish that this new definition of model did not invalidate soundness.

**Proposition 17.** The deductive system  $\mathbf{dupl}_{\uparrow, \downarrow}$  associated to a linear CBPV adjunction with resource and effect modalities in the relative sense of (11) satisfies the rules of  $\mathbf{ILL}^\diamond$ .

Indeed, rephrasing in terms of the oblique morphism distributor, we have a natural family of bijections:

$$\mathcal{O}(!A^-, B^-) \cong \mathcal{M}(Mult A^-, Mult B^-) \tag{12}$$

mapping into  $\mathcal{O}(!A^-, (!B)^-)$ , thus corresponding to the rule  $(\vdash !)$  (with a single antecedent, omitting the lax structure at the moment). Symmetrically for  $\diamond$ , we have a natural family of bijections:

$$\mathcal{O}(A^+, \diamond B^+) \cong \mathcal{C}(FA^+, FB^+) \tag{13}$$

mapping into  $\mathcal{O}((\diamond A)^+, \diamond B^+)$ , thus corresponding to the rule  $(\diamond \vdash)$  (again with a single antecedent, omitting the strength).

In order to be more precise, we will now make use of the notions of (co)algebras for relative (co)monads. Just as for their non-relative counterparts, categories of algebras are terminal in a category of adjoint resolutions for a given relative (co)monad, in such a way that for any relative (co)adjunction there exists a comparison functor factoring the right (resp. left) adjoint (Altenkirch,

Chapman and Uustalu, 2015, Theorem 2.12). The sets (12) thus map via the comparison functor into the set of !-coalgebra morphisms  $\mathcal{V}^!(F^!A^-, F^!B^-)$ , in which values of type  $!A$  are interpreted, where objects  $F^!X^-$  are identified with the types  $!X$  of  $\mathbf{ILL}_p^\diamond$ . Symmetrically, the sets (13) map via the comparison functor into the set of  $\diamond$ -algebra morphisms  $\mathcal{S}^\diamond(G^\diamond A^+, G^\diamond B^+)$  in which covalues of type  $\diamond A$  are interpreted, where objects  $G^\diamond X^+$  are identified with the types  $\diamond X$  of  $\mathbf{ILL}_p^\diamond$ . The rest of the interpretation of  $\mathbf{ILL}_p^\diamond$  into linear CBPV models with resource and effect modalities remains unchanged. Thus:

**Proposition 18.** *The interpretation of  $\mathbf{ILL}^\diamond$  from Proposition 17 extends into a sound interpretation of  $\mathbf{ILL}_p^\diamond$ .*

## 2.5. Examples of $\mathbf{ILL}_p^\diamond$ models

By Proposition 14, any linear CBPV adjunction with (non-relative) resource and effect modalities (3) provides an  $\mathbf{ILL}_p^\diamond$  model. Two series of works, by Jiang, Xue and New (2025) and by Melliès (2025, 2026), motivate a relaxation of the notions of effect and resource modalities to monads relative to  $\uparrow$  and to comonads relative to  $\downarrow$ .

**Example 19.** Jiang, Xue and New (2025) investigate an interface for user-defined programming effects in CBPV using Church encodings and coinductive types. Since universal quantification and coinductive types are typically negative, this leads to several examples of monads whose type of monadic computations is negative, including various examples of implementations of exception, continuation, state and free monads. The situations they study amount to particular  $\mathbf{ILL}_p^\diamond$  models (11) with  $\mathcal{V} = \mathcal{M}$  and  $\mathcal{C} = \text{Alg}_\uparrow(\diamond)$  the category of  $\diamond$ -algebras, i.e. non-linear CBPV models with an effect modality as follows:

$$\begin{array}{ccccc}
 & & F^\diamond & & \\
 & \nearrow & & \searrow & \\
 \mathcal{M} & \xrightarrow{\uparrow} & \mathcal{S} & \xleftarrow{\perp} & \text{Alg}_\uparrow(\diamond) \\
 & \searrow & & \nearrow & \\
 & & G^\diamond & & 
 \end{array} \tag{14}$$

This is a very natural situation owing to the following observation extending the classic results by Beck (1969) on distributive laws between monads:

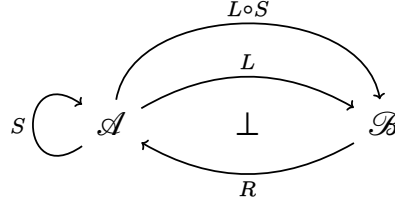
**Definition 20.** Let  $(S, \eta, -^\dagger)$  be a monad on a category  $\mathcal{A}$  and  $L : \mathcal{A} \rightarrow \mathcal{B}$  a functor. A structure of  $L$ -relative monad  $(L \circ S, \tilde{\eta}, -^\ddagger)$  on the functor  $L \circ S : \mathcal{A} \rightarrow \mathcal{B}$  is an *extension* of  $(S, \eta, -^\dagger)$  if and only if for all  $A \in \mathcal{A}$  one has:

$$\begin{aligned}
 \tilde{\eta}_A &= L\eta_A \\
 (Lf)^\ddagger &= L(f^\dagger).
 \end{aligned}$$

**Theorem 21.** *Let  $T, S$  be two monads on a category  $\mathcal{A}$  and  $L : \mathcal{A} \rightleftarrows \mathcal{B} : R$  a resolution of the monad  $T = R \circ L$ . There is an one-to-one correspondence between:*

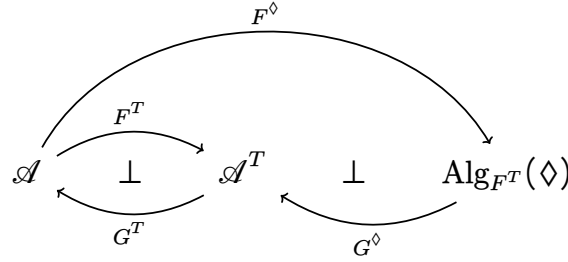
- *distributive laws of monads  $ST \rightarrow TS$ ,*

- extensions of the monad  $S$  to a  $L$ -relative monad structure on  $L \circ S$ , diagrammatically:



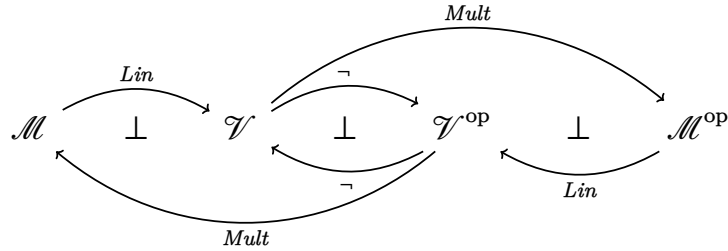
This result is proved in Appendix B.

This theorem generalises the characterisation of distributive laws  $ST \rightarrow TS$ , attributed to Beck, as extensions of  $S$  to the Kleisli category of  $T$ . In particular, given  $T, S$  two monads on a category  $\mathcal{A}$  and a distributive law of monads  $ST \rightarrow TS$ , this result can be applied to the category  $\mathcal{A}^T$  of  $T$ -algebras to provide us with a relative adjoint situation as follows



where  $\Diamond = F^T \circ S$  is the  $F^T$ -relative monad obtained by Theorem 21.

**Example 22.** Melliès (2025, 2026) proposed to extend tensorial logic and dialogue categories to linear logic exponentials, using relative rather than non-relative symmetric monoidal comonads as a way of circumventing obstacles encountered within the programme of *functorial game semantics* (Melliès, 2012a) that are similar to the difficulties we have described previously regarding notions of polarities and focusing for exponentials. This setting corresponds to a variant of the situation (4) involving a dialogue category, now with relative comonad and monad:



**Example 23.** Another approach to a notion of strength for a relative monad is to assume a given monoidal structure on the target category  $\mathcal{S}$ , which then allows us to define the extension of the relative monad under a context  $\Gamma \in \mathcal{S}$ ,

$$\mathcal{S}(\Gamma \otimes \uparrow P, \Diamond Q) \rightarrow \mathcal{S}(\Gamma \otimes \Diamond P, \Diamond Q) \quad (15)$$

This is an assumption made in Liell-Cock, Shirazi and Staton (2026). By way of comparison, we assume that  $\uparrow$  has a right adjoint  $\downarrow$ , which is essential in making our definition of strength (10) work; one advantage in return is that it does not depend on having a monoidal structure on  $\mathcal{S}$ . Such a (strong) monoidal structure on negative types is indeed unavailable in certain situations that interest us, such as categories of algebras in general.

Liell-Cock, Shirazi and Staton (2026) introduce the notion of *RMM-LNL model*, given by a linear-non-linear (LNL) model  $F : \mathcal{M} \rightleftarrows \mathcal{S} : G$  together with a strong  $F$ -relative monad  $\diamond$  in the above sense (15), and some strong monoidal functor from some monoidal category into  $\mathcal{S}$ . Further assuming  $\mathcal{M}$  is distributive, this is again an instance of a non-linear CBPV model with an effect modality (14). Indeed, recall that an LNL model (Benton, 1994) consists in a symmetric monoidal adjunction  $F : \mathcal{M} \rightleftarrows \mathcal{S} : G$  between a cartesian closed category  $\mathcal{M}$  and an symmetric monoidal closed category  $\mathcal{S}$ . It is in particular a (non-linear) CBPV model without sums where  $\uparrow = F$  and  $\downarrow = G$ ; a very particular class in which the category of stacks  $\mathcal{S}$  is monoidal. In an RMM-LNL model, the natural family of morphisms (10) constituting the strength of the  $\uparrow$ -relative monad, instantiated as  $lstr_{\Gamma, P} \in \mathcal{M}(\Gamma \times G\diamond P, G\diamond(\Gamma \times P))$ , derives from

$$\begin{aligned} \eta_{\Gamma \times P} \in \mathcal{S}(F(\Gamma \times P), \diamond(\Gamma \times P)) &\cong \mathcal{S}(F\Gamma \otimes FP, \diamond(\Gamma \times P)) \rightarrow \mathcal{S}(F\Gamma \otimes \diamond P, \diamond(\Gamma \times P)) \rightarrow \\ \mathcal{S}(F\Gamma \otimes FG\diamond P, \diamond(\Gamma \times P)) &\cong \mathcal{S}(F(\Gamma \times G\diamond P), \diamond(\Gamma \times P)) \cong \mathcal{M}(\Gamma \times G\diamond P, G\diamond(\Gamma \times P)). \end{aligned}$$

**Example 24.** By moving from (co)adjunctions to relative (co)adjunctions for modelling resource and effect modalities, we indeed recovered the missing completeness property of the interpretation, giving rise to the syntactic  $\mathbf{ILL}_p^\diamond$  model. This previous syntactic example extends to a more general phenomenon giving rise to a final example of  $\mathbf{ILL}_p^\diamond$  models, obtained by completing another  $\mathbf{ILL}_p^\diamond$  model with its thunkable and linear morphisms. Remember that, in line with the relationships between direct and indirect models of effectful computation (Führmann, 1999; Selinger, 2001), any linear CBPV model  $\uparrow : \mathcal{V} \rightleftarrows \mathcal{S} : \downarrow$  (1) gives rise to a linear CBPV model

$$\begin{array}{ccc} & \uparrow & \\ \mathcal{P}_t & \xrightarrow{\quad} & \mathcal{N}_i \\ & \downarrow & \end{array} \quad \perp \quad (16)$$

obtained starting from the non-associative category  $\mathbf{dupl}_{\uparrow, \downarrow}$  and restricting to positive thunkable and negative linear morphisms. Remember also that this construction is a reflection for the category of adjunctions  $\downarrow \dashv \uparrow$ , and more precisely a completion (of morphisms of  $\mathcal{V}$  into thunkable morphisms, and of morphisms of  $\mathcal{S}$  into linear morphisms of the adjunction) provided the starting adjunction has its unit mono and its counit epi (Munch-Maccagnoni, 2014). Then, resource and effect modalities on  $\mathcal{V} \rightleftarrows \mathcal{S}$ , whether non-relative (3) or relative (11), give rise to relative resource and effect modalities on  $\mathcal{P}_t \rightleftarrows \mathcal{N}_i$ . This is a corollary to the results established in the next section (Corollary 43). In the non-relative case, however, the same obstacle (6) previously mentioned prevents us from obtaining (directly and in general) a non-relative comonad on  $\mathcal{P}_t$  and a non-relative monad on  $\mathcal{N}_i$ .

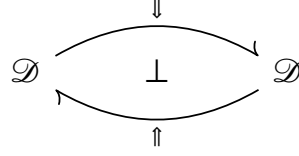
These results motivate in the rest of the paper an investigation from the non-associative perspective into broader notions of model based on resource and effect modalities relative to the linear CBPV adjunction.

### 3. Adjunctions over duploids and relative adjunctions

#### 3.1. Prelude: distributive closed duploids and distributive dialogue duploids

We have recalled previously that every adjunction  $\uparrow \dashv \downarrow$  induces a non-associative category  $\mathbf{dupl}_{\uparrow, \downarrow}$ ; it is in fact a duploid as defined next.

**Definition 25.** An object  $A$  of a non-associative category is **positive** if every morphism  $f : A \rightarrow B$  is linear. It is **negative** if every morphism  $f : B \rightarrow A$  is thunkable. A **duploid** is a non-associative category in which every object is positive or negative (or both), and in which the identity functor has both a positive left adjoint (which we write  $\Downarrow$ ) and a negative right adjoint (written  $\Uparrow$ ). Diagrammatically:



**Definition 26.** An adjunction between two non-associative categories  $F : \mathcal{D} \rightleftarrows \mathcal{E} : G$  (Definition 5) is said to be **positive** when the image of  $F$  on objects is positive. Dually, it is **negative** when the image of  $G$  on objects is negative.

The notions of *symmetric monoidal closed duploid* and of *dialogue duploid*, providing non-associative categorical and axiomatic semantics to **IMLL** (intuitionistic multiplicative linear logic) and **MLL** (classical multiplicative linear logic) respectively, are defined in [MMMM 2026](#). These non-associative categorical and axiomatic semantics are designed to ensure that their equational theory and normal forms coincide with the syntax of the multiplicative fragment of the corresponding  $L$ -calculi.

We now define here the distributive structure in the continuation of Section 1.5.

**Definition 27.** Let  $\mathcal{D}$  be a non-associative category. **Finite sums** on  $\mathcal{D}$  is the structure given by: 1) a positive left adjoint to the diagonal functor  $\Delta : \mathcal{D} \rightarrow \mathcal{D} \times \mathcal{D}$ , which we write  $\oplus$ , 2) a positive left adjoint to the functor  $\mathcal{D} \rightarrow *$  into the trivial category, in other words a positive initial object, which we write  $0$ . **Finite products** (notation  $\&$ ,  $\top$ ) is the structure given by finite sums on  $\mathcal{D}^{\text{op}}$ .

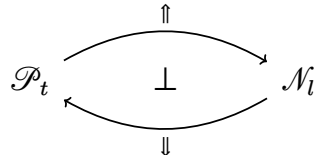
**Definition 28.** A **distributive duploid** is a symmetric monoidal duploid  $\mathcal{D}$  that has finite sums and such that for all  $A, B, C \in \mathcal{D}$  the canonical thunkable map

$$(A \otimes B) \oplus (A \otimes C) \rightarrow A \otimes (B \oplus C)$$

has a thunkable inverse.

### 3.2. A correspondence between relative adjunctions and adjunctions in non-associative direct semantics

Our goal is to extend this non-associative categorical semantics by introducing a notion of linear-non-linear adjunction and strong monad for duploids. One main observation of the present paper is that the appropriate generalisation of linear-non-linear adjunctions to a polarised setting is in correspondence with the notion of relative coadjunction with respect to the polarity shift  $\Downarrow$  appearing as right adjoint in the adjunction associated to the duploid



between the category of positive thunkable maps and the category of negative linear maps, whereas the strong monads on duploids are in correspondence with the notion of relative adjunction with respect to the adjoint polarity shift  $\Uparrow$ .

We first explain how, given an adjunction  $L \dashv R$ , an  $R$ -relative coadjunction  $F \dashv_R G$  induces a positive adjunction on the duploid  $\mathbf{dupl}_{L,R}$  (Proposition 33). By duality, this implies that every  $L$ -relative adjunction  $F \dashv_L G$  induces a negative adjunction on the duploid  $\mathbf{dupl}_{L,R}$ . Conversely, we show that every positive adjunction  $F : \mathcal{C} \rightleftharpoons \mathcal{D} : G$  between an associative category  $\mathcal{C}$  and a duploid  $\mathcal{D}$  induces a  $\Downarrow$ -relative coadjunction (Proposition 34). (And thus dually for the construction of a relative adjunction from negative adjunction.)

Let us recall first a few fundamental and useful properties of adjunctions over non-associative categories, also developed in more detail in [MMMM 2026](#).

**Proposition 29** (R.A.P.L.). *Right adjoints preserve linear maps and their composition.*

**Proposition 30.** *Left adjoints preserve thunkable maps and their composition.*

**Proposition 31.** *The left adjoint  $F : \mathcal{C} \rightarrow \mathcal{D}$  of a positive adjunction where  $\mathcal{C}$  is a category is a functor, and moreover restricts into a functor*

$$F : \mathcal{C} \longrightarrow \mathcal{P}_t$$

*from the category  $\mathcal{C}$  to the category  $\mathcal{P}_t$  of positive thunkable maps in  $\mathcal{D}$ . Dually, the right adjoint  $G : \mathcal{C} \rightarrow \mathcal{D}$  of a negative adjunction where  $\mathcal{C}$  is a category is a functor, and moreover restricts into a functor*

$$G : \mathcal{C} \longrightarrow \mathcal{N}_l$$

*from the category  $\mathcal{C}$  to the category  $\mathcal{N}_l$  of negative linear maps in  $\mathcal{D}$ .*

**Constructing a positive adjunction from an  $R$ -relative coadjunction** To demonstrate the construction of an adjunction on duploids from an  $R$ -relative coadjunction, we start by showing that we can construct a bare functor from the right adjoint of an  $R$ -relative coadjunction.

**Proposition 32.** *Let  $L : \mathcal{A} \rightleftharpoons \mathcal{B} : R$  be an adjunction over associative categories and  $F \dashv_R G : \mathcal{B} \rightarrow \mathcal{C}$  be a  $R$ -relative coadjunction. Then, there exists a bare functor  $G^- : \mathbf{dupl}_{L,R} \rightarrow \mathcal{C}$ .*

*Proof.* On objects, we define  $G^-$  as  $G$  on objects of  $\mathcal{B}$  and as  $G \circ L$  on objects of  $\mathcal{A}$ . Let  $f \in \mathbf{dupl}_{L,R}(X, Y)$ , that is, a morphism in  $\mathcal{A}(X^+, RY^-)$ . The definition of  $G^- f$  depends on whether  $X$  is in  $\mathcal{A}$  or  $\mathcal{B}$ . If  $X$  is in  $\mathcal{A}$ , we define  $G^- f$  as:

$$G^- f \stackrel{\text{def}}{=} G\Phi_{X,Y^-}^{-1}(f) \in \mathcal{C}(GLX, GY^-)$$

where  $\Phi_{A,B} : \mathcal{B}(LA, B) \xrightarrow{\cong} \mathcal{A}(A, RB)$ . If  $X$  is in  $\mathcal{B}$ , we define  $G^-$  as:

$$G^- f \stackrel{\text{def}}{=} \Psi_{GX,Y^-}(f \circ \epsilon_X) \in \mathcal{C}(GX, GY^-)$$

where  $\Psi_{C,B} : \mathcal{A}(FC, RB) \xrightarrow{\cong} \mathcal{C}(C, GB)$ . □

**Proposition 33.** *Every  $R$ -relative coadjunction  $F \dashv_R G$  of the form (9) induces a positive adjunction on  $\mathbf{dupl}_{L,R}$ :*

$$\begin{array}{ccc} & F & \\ \mathcal{C} & \xrightarrow{\quad} & \mathbf{dupl}_{L,R} \\ & \perp & \\ & G^- & \end{array}$$

*Proof.* We construct the bijection of the adjunction  $\mathbf{dupl}_{L,R}(FC, X) \cong \mathcal{C}(C, G^- X)$  by case analysis on  $X$ . If  $X$  is in  $\mathcal{B}$ , then:

$$\mathbf{dupl}_{L,R}(FC, X) = \mathcal{A}(FC, RX) \cong \mathcal{C}(C, GX) \cong \mathcal{C}(C, G^- X)$$

If  $X$  is in  $\mathcal{A}$ , then:

$$\mathbf{dupl}_{L,R}(FC, X) = \mathcal{A}(FC, RLX) \cong \mathcal{C}(C, GLX) \cong \mathcal{C}(C, G^- X) \quad \square$$

### Constructing a $\Downarrow$ -relative coadjunction from a positive adjunction

**Proposition 34.** *Every positive adjunction  $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$  where  $\mathcal{C}$  is a category induces a  $\Downarrow$ -relative coadjunction of the form:*

$$\begin{array}{ccc} & \mathcal{C} & \\ F \swarrow & \perp & \nwarrow G \\ \mathcal{P}_t & \xleftarrow{\Downarrow} & \mathcal{N}_l \end{array}$$

where we write  $G$  for the restriction of the original right adjoint  $G : \mathcal{D} \rightarrow \mathcal{C}$ .

*Proof.* Let  $C$  be an object of  $\mathcal{C}$  and  $N$  an object of  $\mathcal{N}_l$ . We construct the bijection of the  $\Downarrow$ -relative adjunction by using the bijection of the adjunction on duploids:

$$\mathcal{P}_t(FC, \Downarrow N) \cong \mathcal{D}(FC, N) \cong \mathcal{C}(C, GN) \quad \square$$

### 3.3. Relative LNL coadjunctions and direct LNL adjunctions

We now consider the case where the modality is the exponential  $A \mapsto !A$ . The linear-non-linear adjunction becomes, when polarized, either a  $R$ -relative linear-non-linear coadjunction in an indirect setting or a positive adjunction with a cartesian category in a direct setting. We also show how we can construct one from the other.

**Definition 35.** Let  $\mathcal{A}$  and  $\mathcal{B}$  be two categories such that  $\mathcal{A}$  is symmetric monoidal category and  $R : \mathcal{B} \rightarrow \mathcal{A}$  be a functor. A  **$R$ -relative linear-non-linear coadjunction** is a  $R$ -relative coadjunction:

$$\begin{array}{ccccc} & & \text{Lin} & & \\ & \searrow & & \swarrow & \\ \mathcal{M} & & \perp & & \mathcal{A} \\ & \swarrow & & \searrow & \\ & & \text{Mult} & & \mathcal{B} \end{array} \quad (17)$$

such that  $\mathcal{M}$  is cartesian and  $\text{Lin}$  is a strong monoidal functor.

This relative coadjunction means that we have a family of bijections :

$$\varphi_{\bar{X}_i, A} : \mathcal{A}(\text{Lin } X_1 \otimes \cdots \otimes \text{Lin } X_n, RA) \cong \mathcal{M}(X_1 \times \cdots \times X_n, \text{Mult } A)$$

natural in  $X_1, \dots, X_n$  and  $A$ .

**Definition 36.** A **direct linear-non-linear adjunction** is a positive adjunction

$$\begin{array}{ccc} & \text{Lin} & \\ \mathcal{M} & \xrightarrow{\quad} & \mathcal{D} \\ & \text{Mult} & \end{array}$$

between a symmetric monoidal duploid  $\mathcal{D}$  and a cartesian category  $\mathcal{M}$  such that the functor from  $\mathcal{M}$  to  $\mathcal{P}_t$  induced by  $\text{Lin}$  is strong monoidal.

### Correspondence theorem

**Proposition 37.** *Every  $R$ -relative linear-non-linear coadjunction  $Lin \dashv_R Mult$  of the form (17) induces a direct linear-non-linear adjunction:*

$$\begin{array}{ccc} & Lin & \\ \mathcal{M} & \xrightarrow{\quad} & \mathbf{dupl}_{L,R} \\ & \xleftarrow{Mult^-} & \end{array} \quad \perp \quad (18)$$

**Proposition 38.** *Every direct linear-non-linear adjunction  $Lin : \mathcal{M} \rightleftharpoons \mathcal{D} : Mult$  induces a  $\Downarrow$ -relative linear-non-linear coadjunction of the form:*

$$\begin{array}{ccc} & \mathcal{M} & \\ Lin \swarrow & \dashv & \searrow Mult \\ \mathcal{P}_t & \Downarrow & \mathcal{N}_l \end{array}$$

### 3.4. Strength for a relative monad with respect to a comonad

Finally, we consider the case where we have the resource modality  $A \mapsto !A$  and an effect modality  $A \mapsto \Diamond A$  strong with respect to  $!$ .

**Definition 39.** Suppose given an adjunction  $L : \mathcal{A} \rightleftharpoons \mathcal{B} : R$  where the category  $\mathcal{A}$  is equipped with a symmetric monoidal structure  $(\otimes, 1)$  and where the monad  $R \circ L$  is strong. Suppose also given an  $L$ -relative adjunction  $F \dashv G$  as well as an  $R$ -relative linear-non-linear coadjunction  $Lin \dashv Mult$ . Diagrammatically:

$$\begin{array}{ccccc} & & F & & \\ & & \curvearrowright & & \\ \mathcal{M} & \xrightarrow{Lin} & \mathcal{A} & \xrightarrow{L} & \mathcal{B} & \xrightarrow{F} & \mathcal{C} \\ & \xleftarrow{Mult} & \xleftarrow{R} & \xleftarrow{G} & \\ & & \perp & & \end{array} \quad (19)$$

The relative monad  $\Diamond = G \circ F$  is strong with respect to the functor  $Lin$  if it is equipped with a natural family of morphisms in  $\mathcal{A}$

$$str_{X,A} : Lin X \otimes R\Diamond A \rightarrow R\Diamond(Lin X \otimes A)$$

making the four usual coherence diagrams commute.

**Definition 40.** Let a direct linear-non-linear adjunction  $Lin : \mathcal{M} \rightleftharpoons \mathcal{D} : Mult$  and a negative adjunction  $F : \mathcal{D} \rightleftharpoons \mathcal{C} : G$ . Diagrammatically:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{Lin} & \mathcal{D} & \xrightarrow{F} & \mathcal{C} \\ & \xleftarrow{Mult} & \xleftarrow{G} & \\ & & \perp & \end{array} \quad (20)$$

The bare endofunctor  $\diamond = G \circ F$  is strong with respect to the bare functor  $Lin$  if it is equipped with a natural family of morphisms in  $\mathcal{D}$

$$str_{X,A} : Lin X \otimes \diamond A \rightarrow \diamond(Lin X \otimes A)$$

linear wrt.  $\diamond A$  (MMMM 2026, Definition F.2, p.42), making the four usual coherence diagrams commute.

## Correspondence theorem

**Proposition 41.** *Every situation of the form (19) such that the  $L$ -relative monad  $G \circ F$  is strong with respect to  $Lin$  induces a negative adjunction:*

$$\begin{array}{ccc} & F^+ & \\ \text{dupl}_{L,R} & \xrightarrow{\quad} & \mathcal{C} \\ & G & \end{array} \quad \perp$$

such that  $G \circ F^+$  is strong with respect to  $Lin$ , the left adjoint of the induced direct linear-non-linear adjunction from (18).

**Proposition 42.** *Every situation of the form (20) such that the bare endofunctor  $G \circ F$  is strong with respect to  $Lin$  induces a  $\uparrow$ -relative adjunction:*

$$\begin{array}{ccc} & \mathcal{C} & \\ F \nearrow & \dashv & \searrow G \\ \mathcal{P}_t & \xrightarrow{\quad} & \mathcal{N}_l \\ & \uparrow & \end{array}$$

such that the relative monad  $G \circ F$  is strong with respect to  $Lin$ .

**Corollary 43.** *Any  $\mathbf{ILL}_p^\diamond$  model over an adjunction  $\uparrow : \mathcal{V} \rightleftarrows \mathcal{S} : \downarrow$  induces an  $\mathbf{ILL}_p^\diamond$  model over its thunkable-linear completion  $\uparrow : \mathcal{P}_t \rightleftarrows \mathcal{N}_l : \downarrow$ .*

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## Abbreviations

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## A. Appendix

### A.1. Componentwise bare and proper functors

A componentwise bare functor

$$F : \mathcal{A}, \mathcal{B} \longrightarrow \mathcal{C}$$

between non associative categories  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{C}$ , is a pair consisting of a function

$$F_{\text{obj}} : \mathcal{A}_{\text{obj}} \times \mathcal{B}_{\text{obj}} \longrightarrow \mathcal{C}_{\text{obj}}$$

which associates an object  $F(A, B)$  of  $\mathcal{C}$  to every pair  $(A, B)$  consisting of an object  $A$  of  $\mathcal{A}$  and of an object  $B$  of  $\mathcal{B}$ ; together with a family of functions

$$F_{\text{fst}}^B(A_1, A_2) : \mathcal{A}(A_1, A_2) \longrightarrow \mathcal{C}((A_1, B), (A_2, B))$$

$$F_{\text{snd}}^A(B_1, B_2) : \mathcal{B}(B_1, B_2) \longrightarrow \mathcal{C}((A, B_1), (A, B_2))$$

such that the series of equalities holds:

$$F_{\text{fst}}^B(\text{id}A) = \text{id}_{F_{\text{obj}}(A, B)} \quad F_{\text{snd}}^A(\text{id}B) = \text{id}_{F_{\text{obj}}(A, B)}$$

A componentwise proper functor is a componentwise bare functor such that the equality holds:

$$F_{\text{fst}}^B(g \circ f) = F_{\text{fst}}^B(g) \circ F_{\text{fst}}^B(f)$$

$$F_{\text{snd}}^A(g \circ f) = F_{\text{snd}}^A(g) \circ F_{\text{snd}}^A(f)$$

### A.2. Tensor product

The tensor product  $\mathcal{A} \boxtimes \mathcal{B}$  of two non associative categories  $\mathcal{A}$  and  $\mathcal{B}$  is the non associative category whose objects are pairs  $(A, B)$

$$(A, g) : (A, B_1) \longrightarrow (A, B_2)$$

$$(f, B) : (A_1, B) \longrightarrow (A_2, B)$$

such that

$$(A, g_2 \circ g_1) = (A, g_2) \circ (A, g_1) \quad (A, \text{id}B) = \text{id}(A, B)$$

$$(f_2 \circ f_1, B) = (f_2, B) \circ (f_1, B) \quad (\text{id}A, B) = \text{id}(A, B)$$

There is a componentwise proper functor

$$\mathcal{A}, \mathcal{B} \longrightarrow \mathcal{A} \boxtimes \mathcal{B}$$

which induces a bijection between componentwise proper functors

$$\mathcal{A}, \mathcal{B} \longrightarrow \mathcal{C}$$

and proper functors

$$\mathcal{A} \boxtimes \mathcal{B} \longrightarrow \mathcal{C}$$

Moreover, it induces a bijection between componentwise bare functors

$$\mathcal{A}, \mathcal{B} \longrightarrow \mathcal{C}$$

and bare functors

$$\mathcal{A} \boxtimes \mathcal{B} \longrightarrow \mathcal{C}$$

## B. Proof of Theorem 21

*Proof.* An extension of  $S$  to a  $L$ -relative monad structure on  $L \circ S$  amounts to  $\tilde{\eta}, -^*$  satisfying:

1.  $\tilde{\eta}_A = L\eta_A^S$  for all  $A$
2.  $(Lf)^* = L(\mu_B^S \circ Sf)$  for all  $f : A \rightarrow SB$
3.  $f = f^* \circ \tilde{\eta}_A$  for all  $f : A \rightarrow SB$
4.  $(f^* \circ g)^* = f^* \circ g^*$  for all  $f : LA \rightarrow LSB$  and  $g : LB \rightarrow LSC$

In particular, 1. and 2. above imply  $\tilde{\eta}_A^* = \text{id}_{LSA}$ .

We note  $\varphi : \mathcal{B}(L-, =) \rightarrow \mathcal{A}(-, R=)$  (and  $\varphi = \varphi_{A,B}$  by abuse of notation).

( $\Rightarrow$ ) Let  $\lambda : ST \rightarrow TS$  be a distributive law of monads. 1. constrains the definition

$$\tilde{\eta} = L\eta^S$$

We define  $f^*$  for  $f : LA \rightarrow LSA$  with:

$$\varphi(f^*) = T\mu_B^S \circ \lambda_{SB} \circ S\varphi(f)$$

We show 2-4. above of which 2. and 3. are rather straightforward. For 2., one has:

$$\begin{aligned} \varphi((Lf)^*) &= T\mu_B^S \circ \lambda_{SB} \circ S\varphi(Lf) \\ &= T\mu_B^S \circ \lambda_{SB} \circ S(\eta_{SB}^T \circ f) \\ &= T\mu_B^S \circ \eta_{SSB}^T \circ Sf \\ &= \eta_{SB}^T \circ \mu_B^S \circ Sf \end{aligned}$$

hence the result. For 3., one has:

$$\begin{aligned} \varphi(f^* \circ L\eta_A^S) &= \varphi(f^*) \circ \eta_A^S \\ &= T\mu_B^S \circ \lambda_{SB} \circ S\varphi(f) \circ \eta_A^S \\ &= T\mu_B^S \circ \lambda_{SB} \circ \eta_{TSB}^S \circ \varphi(f) \\ &= T\mu_B^S \circ T\eta_B^S \circ \varphi(f) \\ &= \varphi(f) \end{aligned}$$

For 4., we show the equality of  $\varphi((f^* \circ g)^*)$  and  $\varphi(f^* \circ g^*)$  as follows:

$$\begin{aligned} \varphi((f^* \circ g)^*) &= T\mu_C^S \circ \lambda_{SC} \circ S(\mu_{SC}^T \circ T(T\mu_C^S \circ \lambda_{SC} \circ Sf')) \circ Sg' \\ &= T\mu_C^S \circ \lambda_{SC} \circ ST\mu_C^S \circ S(\mu_{SSC}^T \circ T(\lambda_{SC} \circ Sf')) \circ Sg' \\ &= T\mu_C^S \circ TS\mu_C^S \circ \lambda_{SSC} \circ S(\mu_{SSC}^T \circ T(\lambda_{SC} \circ Sf')) \circ Sg' \\ &= T\mu_C^S \circ TS\mu_C^S \circ \mu_{SSSC}^T \circ T\lambda_{SSC} \circ \lambda_{TSSC} \circ ST(\lambda_{SC} \circ Sf') \circ Sg' \\ &= T\mu_C^S \circ TS\mu_C^S \circ \mu_{SSSC}^T \circ T\lambda_{SSC} \circ TS(\lambda_{SC} \circ Sf') \circ \lambda_{SB} \circ Sg' \\ &= T\mu_C^S \circ \underbrace{T\mu_{SC}^S \circ \mu_{SSSC}^T \circ T\lambda_{SSC} \circ TS(\lambda_{SC} \circ Sf')}_a \circ \lambda_{SB} \circ Sg' \\ \varphi(f^* \circ g^*) &= T\mu_C^S \circ \underbrace{\mu_{SSC}^T \circ T(\lambda_{SC} \circ Sf') \circ T\mu_B^S}_{b} \circ \lambda_{SB} \circ Sg' \end{aligned}$$

where

$$\begin{aligned}
a &= T\mu_{SC}^S \circ \mu_{SSC}^T \circ T\lambda_{SSC} \circ TS(\lambda_{SC} \circ Sf') \\
&= \mu_{SSC}^T \circ TT\mu_{SC}^S \circ T\lambda_{SSC} \circ TS(\lambda_{SC} \circ Sf') \\
&= \mu_{SSC}^T \circ T(T\mu_{SC}^S \circ \lambda_{SSC} \circ S\lambda_{SC} \circ SSf') \\
&= \mu_{SSC}^T \circ T(\lambda_{SC} \circ \mu_{TSC}^S \circ SSf') \\
&= \mu_{SSC}^T \circ T(\lambda_{SC} \circ Sf' \circ \mu_B^S) = b
\end{aligned}$$

( $\Leftarrow$ ) It is sufficient to show that the relative monad defines an extension  $(\hat{S}, \hat{\eta}, \hat{\mu})$  of the monad  $S$  to the Kleisli category  $\mathcal{A}_T$  of  $T$ , that is to say a monad such that:

$$\begin{aligned}
\hat{S} \circ F_T &= F_T \circ S \\
\hat{\eta}_A &= F_T \eta_A^S = \eta_A^T \circ_{\mathcal{A}} \eta_A^S \\
\hat{\mu}_A &= F_T \mu_A^S = \eta_A^T \circ_{\mathcal{A}} \mu_A^S
\end{aligned}$$

(See for instance Hyland, Nagayama et al. 2006, Section 3.) Note that these conditions necessarily make  $(\hat{S}, \hat{\eta}, \hat{\mu})$  a monad on  $\mathcal{A}_T$ . Note also that a morphism  $f \in \mathcal{B}(LA, LB)$  is essentially the same thing as a morphism  $\varphi(f) \in \mathcal{A}_T(A, B)$ . We define in this way:

$$\begin{aligned}
\text{for all } \varphi(f) \in \mathcal{A}_T(A, B): \quad \hat{S}\varphi(f) &= \varphi((\tilde{\eta}_B \circ f)^*) \\
\hat{\eta}_A &= \varphi(\tilde{\eta}_A) \\
\hat{\mu}_A &= \varphi((\text{id}_{LSA})^*)
\end{aligned}$$

Now the second and third conditions of a Kleisli extension amount by adjunction to

$$\begin{aligned}
\hat{\eta}_A &= \varphi(L\eta_A^S) \\
\hat{\mu}_A &= \varphi(L\mu_A^S)
\end{aligned}$$

which are immediate by 1. and 2. Finally the first condition also holds as follows:

$$\begin{aligned}
\hat{S}F_T f^{A \rightarrow B} &= \hat{S}(\varphi(Lf)) \\
&= \varphi((\tilde{\eta}_B \circ Lf)^*) \\
&= \varphi((L\eta_B^S \circ Lf)^*) \\
&= \varphi(L(\mu_B^S \circ S(\eta_B^S \circ f))) \\
&= \varphi(LSf) \\
&= F_T Sf
\end{aligned}$$

□

## C. IMALL<sub>p</sub> has sums

**Definition 44.** A non-associative category  $\mathcal{D}$  has (binary) sums if  $\Delta$  has a positive left adjoint, in other words a bare functor  $\oplus : \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}$  such that  $A \oplus B$  is positive and such that there exists a family of isomorphisms:

$$\mathcal{D}(A \oplus B, C) \cong \mathcal{D}(A, C) \times \mathcal{D}(B, C)$$

which is natural in  $(\mathcal{D} \times \mathcal{D})^{\text{op}} \boxtimes \mathcal{D} \rightarrow \text{Set}$ .

Let us spell out the naturality condition more explicitly: for any  $f : A \rightarrow C$  and  $g : B \rightarrow C$  there exists a pairing  $[f, g] : A \oplus B \rightarrow C$  such that:

$$\forall h : C \rightarrow D, \quad h \circ [f, g] = [h \circ f, h \circ g] \quad (21)$$

$$\forall h_1 : A' \rightarrow A, \forall h_2 : B' \rightarrow B, \quad [f, g] \circ (h_1 \oplus h_2) = [f \circ h_1, g \circ h_2] \quad (22)$$

**Proposition 45.** *The syntactic duploid has sums given by the type  $\oplus$  and its constructors:*

$$\begin{aligned} \oplus_{A_x, B_y, C, D, z} &: \mathcal{D}(A_x, C) \times \mathcal{D}(B_y, D) \rightarrow \mathcal{D}((A \oplus B)_z, C \oplus D) \\ t \oplus_{x, y, z} u &\stackrel{\text{def}}{=} \mu\alpha. \langle z \parallel \bar{\mu}[x. \langle \iota_1(t) \parallel \alpha \rangle \mid y. \langle \iota_2(u) \parallel \alpha \rangle] \rangle \\ \text{where } \iota_i(s) &\stackrel{\text{def}}{=} \mu\alpha. \langle t \parallel \bar{\mu}x. \langle \iota_i(x) \parallel \alpha \rangle \rangle \\ \varphi_{A, B, C} &: \begin{cases} \mathcal{D}(A \oplus B, C_\alpha) \rightarrow \mathcal{D}(A, C_\alpha) \times \mathcal{D}(B, C_\alpha) \\ e \mapsto (\bar{\mu}x. \langle \iota_1(x) \parallel e \rangle, \bar{\mu}y. \langle \iota_2(y) \parallel e \rangle) \end{cases} \\ \psi_{A, B, C} &: \begin{cases} \mathcal{D}(A, C_\alpha) \times \mathcal{D}(B, C_\alpha) \rightarrow \mathcal{D}(A \oplus B, C_\alpha) \\ (e_1, e_2) \mapsto [e_1, e_2] \stackrel{\text{def}}{=} \bar{\mu}[x. \langle x \parallel e_1 \rangle \mid y. \langle y \parallel e_2 \rangle] \end{cases} \end{aligned}$$

*Proof.*

- Positivity: (by definition, and in fact) from the observation that expansion at type  $A \oplus B$  implies the linearity of all  $e \in \mathcal{D}(A \oplus B, C)$ .
- $(\psi \circ \varphi)_{A, B, C} = \text{id}_{\mathcal{D}(A \oplus B, C)}$  amounts to the equality between terms:

$$\begin{aligned} &\bar{\mu}[x. \langle x \parallel \bar{\mu}x. \langle \iota_1(x) \parallel e \rangle \rangle \mid y. \langle y \parallel \bar{\mu}y. \langle \iota_2(y) \parallel e \rangle \rangle] \\ &\simeq_R \bar{\mu}[x. \langle \iota_1(x) \parallel e \rangle \mid y. \langle \iota_2(y) \parallel e \rangle] \\ &\simeq_E e \end{aligned}$$

- $(\varphi \circ \psi)_{A, B, C} = \text{id}_{\mathcal{D}(A, C) \times \mathcal{D}(B, C)}$  amounts to the equality between terms:

$$\begin{aligned} &\bar{\mu}y. \langle \iota_i(y) \parallel \bar{\mu}[x_1. \langle x_1 \parallel e_1 \rangle \mid x_2. \langle x_2 \parallel e_2 \rangle] \rangle \\ &\simeq_R \bar{\mu}y. \langle y \parallel e_i \rangle \\ &\simeq_e e_i \end{aligned}$$

- The naturality condition (21) amounts, for all  $e_1 \in \mathcal{D}(A, C_\alpha)$ ,  $e_2 \in \mathcal{D}(B, C_\alpha)$  and  $e_3 \in \mathcal{D}(C, D)$ , to the equality between terms:

$$\begin{aligned} c_1 &\stackrel{\text{def}}{=} \langle \mu\alpha. \langle x \parallel \bar{\mu}[y. \langle y \parallel e_1 \rangle \mid z. \langle z \parallel e_2 \rangle] \rangle \parallel e_3 \rangle \\ c_2 &\stackrel{\text{def}}{=} \langle x \parallel \bar{\mu}[y. \langle \mu\alpha. \langle y \parallel e_1 \rangle \parallel e_3 \rangle \mid z. \langle \mu\alpha. \langle z \parallel e_2 \rangle \parallel e_3 \rangle] \rangle \\ c_1 &\simeq_{\text{RE}} c_2 \end{aligned}$$

when  $e_3$  is not necessarily linear. It holds thanks to  $\triangleright_E$ :

$$\begin{aligned} &\bar{\mu}x. c_1 \triangleright_E \bar{\mu}[y. \langle \iota_1(y) \parallel \bar{\mu}x. c_1 \rangle \mid z. \langle \iota_2(z) \parallel \bar{\mu}x. c_1 \rangle] \\ &\rightarrow_R^* \bar{\mu}[y. \langle \mu\alpha. \langle y \parallel e_1 \rangle \parallel e_3 \rangle \mid z. \langle \mu\alpha. \langle z \parallel e_2 \rangle \parallel e_3 \rangle] \\ &\triangleleft_r \bar{\mu}x. \langle x \parallel \bar{\mu}[y. \langle \mu\alpha. \langle y \parallel e_1 \rangle \parallel e_3 \rangle \mid z. \langle \mu\alpha. \langle z \parallel e_2 \rangle \parallel e_3 \rangle] \rangle \\ &= \bar{\mu}x. c_2 \end{aligned}$$

- The naturality condition (22) amounts, for all  $e_1 \in \mathcal{D}(A, C_\alpha)$ ,  $e_2 \in \mathcal{D}(B, C_\alpha)$ ,  $t \in \mathcal{D}(A'_x, A)$ ,  $u \in \mathcal{D}(B'_y, B)$ , to the equality between terms:

$$\begin{aligned} c_1 &\stackrel{\text{def}}{=} \langle t \oplus_{x,y,z} u \parallel [e_1, e_2] \rangle \\ c_2 &\stackrel{\text{def}}{=} \langle z \parallel \bar{\mu}[x.\langle t \parallel e_1 \rangle \mid y.\langle u \parallel e_2 \rangle] \rangle \\ c_1 &\simeq_{\text{RE}} c_2 \end{aligned}$$

It holds indeed:

$$\begin{aligned} \langle t \oplus_{x,y,z} u \parallel [e_1, e_2] \rangle &= \langle \mu\alpha.\langle z \parallel \bar{\mu}[x.\langle \iota_1(t) \parallel \alpha \rangle \mid y.\langle \iota_2(u) \parallel \alpha \rangle] \parallel [e_1, e_2] \rangle \\ &\triangleright_r \langle z \parallel \bar{\mu}[x.\langle \iota_1(t) \parallel [e_1, e_2] \rangle \mid y.\langle \iota_2(u) \parallel [e_1, e_2] \rangle] \rangle \\ &\rightarrow^* \langle z \parallel \bar{\mu}[x.\langle t \parallel e_1 \rangle \mid y.\langle u \parallel e_2 \rangle] \rangle \\ &= c_2 \end{aligned} \quad \square$$

## D. Relative monads and comonads in $\mathbf{ILL}_p^\diamond$

We prove that the syntactic linear CBPV model  $\mathcal{V} \rightleftharpoons \mathcal{S}$  of thunkable and linear terms of  $\mathbf{ILL}_p^\diamond$  has a relative comonad  $!$  on  $\downarrow : \mathcal{S} \rightarrow \mathcal{V}$ . A thunkable map  $!N \vdash \downarrow M$  of the syntactic model is the same thing as a term with judgement  $!N \vdash M$  in  $\mathbf{ILL}_p^\diamond$ . The co-unit  $\varepsilon_N$  is thus given by the following covalue:

$$\mid !\alpha : !N \vdash \alpha : N$$

Given any thunkable map  $!N \vdash \downarrow M$ , in the form of a command

$$c : (x : !N \vdash \alpha : M),$$

its coextension  $^*c$  is the value:

$$x : !N \vdash \underbrace{\mu!\alpha.c}_{^*c} : !M$$

which is indeed thunkable by definition. The two unital laws

$$\varepsilon \circ ^*f = f \qquad \text{id}_{SB} = ^*\varepsilon_B$$

coincide respectively with the following instances of reduction and expansion:

$$\langle \mu!\alpha.c \parallel !\alpha \rangle^+ \triangleright_R c \qquad x \triangleright_E \mu!\alpha.\langle x \parallel !\alpha \rangle$$

The associativity law:

$$^*(f \circ ^*g) = ^*f \circ ^*g$$

instantiated for  $c : (x : !N_1 \vdash \alpha : N_2)$  and  $c' : (y : !N_2 \vdash \beta : N_3)$  corresponds to

$$\mu!\beta.\langle \mu!\alpha.c \parallel \bar{\mu}y^+.c' \rangle^+ \simeq_{\text{RE}} \mu\gamma^+.\langle \mu!\alpha.c \parallel \bar{\mu}y^+.\langle \mu!\beta.c' \parallel \gamma \rangle^+ \rangle^+$$

which indeed holds given that both sides are convertible to  $\mu!\beta.c'[\mu!\alpha.c/y]$ .

The case of the relative monad  $\diamond : \mathcal{V} \rightarrow \mathcal{S}$  on  $\uparrow : \mathcal{V} \rightarrow \mathcal{S}$  is symmetric. This establishes Proposition 16.

**Proposition 46.** *A term  $x : !A \vdash t : !B$  is a  $!$ -coalgebra morphism in the syntactic model if and only if the following commutation holds for any  $c$ :*

$$\langle t \parallel \bar{\mu}y^+.\langle \mu!\beta.c \parallel \alpha \rangle^+ \rangle^+ \simeq_{\text{RE}} \langle \mu!\beta.\langle t \parallel \bar{\mu}y^+.c \rangle^+ \parallel \alpha \rangle^+$$

*Dually, a term  $e : \Diamond A \vdash \alpha : \Diamond B$  is a  $\Diamond$ -algebra morphism in the syntactic model if and only if the following commutation holds for any  $c$ :*

$$\langle \mu\beta^-\langle x \parallel \bar{\mu}\Diamond y.c \rangle^- \parallel e \rangle^- \simeq_{\text{RE}} \langle x \parallel \bar{\mu}\Diamond y.\langle \mu\beta^-.c \parallel e \rangle^- \rangle^-.$$

These characterisations extend beyond free (co)algebras if we add other types of (co)algebras to the type system making these commutations well typed.

## E. Focused normal forms for $\mathbf{ILL}_p^\Diamond$

### E.1. Strong normalisation

This section is taken from (Munch-Maccagnoni, 2017) with minor adaptations, which was itself inspired by Krivine's adaptation (Krivine, 1993, 2009) of Girard's reducibility candidates (Girard, 1972, 1987).

Judgements refer to typability in  $\mathbf{ILL}_p^\Diamond$  (or any extension with additional structural rules, since the restriction on structural rules is not accounted for by the model).

**Proposition 47** (Weak standardization). *If  $c (\rightarrow_R \setminus \triangleright_R)^* c' \triangleright_R c''$  then there exists  $c'''$  such that  $c \triangleright_R c''' \rightarrow_R^* c''$ .*

*Proof.* Suppose one has  $c (\rightarrow_R \setminus \triangleright_R)^* c' \triangleright_R c''$ . We consider the rule involved in the reduction  $c' \triangleright_R c''$ : if it is  $(R\otimes)$  then  $c'$  is of the form  $\langle V' \otimes W' \parallel \bar{\mu}(x \otimes y).c_0' \rangle$  and  $c'' = c_0'[V'/x, W'/y]$ . Now  $\rightarrow_R \setminus \triangleright_R$  can only create strict subcommands, therefore  $c$  is of the form  $\langle V \otimes W \parallel \bar{\mu}(x \otimes y).c_0 \rangle$  where  $V \rightarrow_R^* V'$ ,  $W \rightarrow_R^* W'$ , and  $c_0 \rightarrow_R^* c_0'$ . Therefore by taking  $c''' = c_0[V/x, W/y]$  one has  $c \triangleright_R c''' \rightarrow_R^* c_0'$ . Similarly for all other reduction rules: no redex for  $\triangleright_R$  can be created by  $\rightarrow_R \setminus \triangleright_R$  and one can apply the same reasoning.  $\square$

**Definition 48.** A command  $c$  is strongly normalizing ( $c \in \perp\!\!\!\perp$ ) if any  $\rightarrow_R$ -sequence starting from  $c$  is finite. An expression  $t$  (respectively a context  $e$ ) is strongly normalizing ( $t \in \mathbb{T}$ , resp.  $e \in \mathbb{E}$ ) if any  $\langle t \parallel \alpha \rangle^\varepsilon$  (resp. any  $\langle x \parallel e \rangle^\varepsilon$ ) is in  $\perp\!\!\!\perp$ .

**Lemma 49.** *Let  $c$  be a command. If all strict sub-commands of  $c$  are in  $\perp\!\!\!\perp$ , and if either  $c \not\triangleright_R$  or  $c \triangleright_R c' \in \perp\!\!\!\perp$  for some command  $c'$ , then  $c \in \perp\!\!\!\perp$ .*

*Proof.* Assume that  $c$  admits an infinite  $\rightarrow_R$ -sequence. If this sequence is an infinite  $(\rightarrow_R \setminus \triangleright_R)$ -sequence, then at least one immediate sub-command of  $c$  (there are finitely many so) admits an infinite  $\rightarrow_R$ -sequence. Otherwise, let us write  $c' \triangleright_R c''$  the first occurrence of  $\triangleright_R$  in the sequence. By Proposition 47 there exists  $c'''$  with  $c \triangleright_R c''' \rightarrow_R^* c''$  and therefore  $c'''$  admits an infinite  $\rightarrow_R$ -sequence. In this case, since  $\triangleright_R$  is deterministic, this means that all  $\triangleright_R$ -reducts admit an infinite  $\rightarrow_R$ -sequence.  $\square$

**Lemma 50.** *One has  $t \in \mathbb{T}$  (respectively  $e \in \mathbb{E}$ ) if and only if all sub-commands of  $t$  (resp.  $e$ ) are in  $\perp\!\!\!\perp$ .*

*Proof.*  $(\Rightarrow)$  Immediate.  $(\Leftarrow)$  Assume that all sub-commands of  $t$  are in  $\perp\!\!\!\perp$ . We apply Lemma 49 on the command  $\langle t \parallel \alpha \rangle^\varepsilon$ . If there exists  $c$  with  $\langle t \parallel \alpha \rangle^\varepsilon \triangleright_R c$ , then one has  $t = \mu\beta^\varepsilon.c'$  with  $c = c'[\alpha/\beta]$ . Now  $c'$ , being a sub-command of  $t$ , is in  $\perp\!\!\!\perp$ , and since the rewriting rules are left-linear, variable substitution cannot unlock new reductions. Therefore  $c \in \perp\!\!\!\perp$  as required. The proof is similar for  $e$ .  $\square$

Notice that the direction  $(\Leftarrow)$  relies on eliminations (e.g. function application  $tu$ ) being expressed with a  $\mu$  binder. In calculi where  $tu$  is given as a primitive, it is not possible to state that  $\langle t \parallel u \cdot \alpha \rangle$  is a sub-command of  $tu$ !

**Proposition 51** (Saturation). *One has  $\langle t \parallel e \rangle \in \perp\!\!\!\perp$  if and only if  $t \in \mathbb{T}$ ,  $e \in \mathbb{E}$ , and either  $\langle t \parallel e \rangle \not\triangleright_R$  or  $\langle t \parallel e \rangle \triangleright_R c \in \perp\!\!\!\perp$ .*

*Proof.* By Lemma 49 and Lemma 50.  $\square$

As usual, we let  $\perp\!\!\!\perp$  define an antitone Galois correspondence  $\cdot^\perp \dashv \cdot^\perp : \mathcal{P}(\mathbb{T}) \rightarrow \mathcal{P}(\mathbb{E})^{\text{op}}$ . In fact, we define two such Galois correspondences:

$$\begin{aligned} (\cdot^{\perp+}) \dashv (\cdot^{\perp+}) : \mathcal{P}(\mathbb{T}) &\rightarrow \mathcal{P}(\mathbb{S})^{\text{op}} \\ (\cdot^{\perp-}) \dashv (\cdot^{\perp-}) : \mathcal{P}(\mathbb{V}) &\rightarrow \mathcal{P}(\mathbb{E})^{\text{op}} \end{aligned}$$

by the following two poles:

$$\begin{aligned} t \perp\!\!\!\perp^+ S &\iff \langle t \parallel S \rangle^+ \in \perp\!\!\!\perp \\ V \perp\!\!\!\perp^- e &\iff \langle V \parallel e \rangle^- \in \perp\!\!\!\perp \end{aligned}$$

**Definition 52.**

- $\mathbb{V} \subseteq \mathbb{T}$  is the set of strongly normalising values.
- $\mathbb{S} \subseteq \mathbb{E}$  is the set of strongly normalising stacks.
- We write  $\mathcal{X}_{\mathbb{V}} \stackrel{\text{def}}{=} \mathcal{X} \cap \mathbb{V}$  and  $\mathcal{X}_{\mathbb{S}} \stackrel{\text{def}}{=} \mathcal{X} \cap \mathbb{S}$ .
- We consider the following maps:

$$\begin{aligned} \mathcal{V} \otimes \mathcal{W} &\stackrel{\text{def}}{=} \{V \otimes W \mid V \in \mathcal{V} \text{ and } W \in \mathcal{W}\} \\ \mathcal{V}_1 \oplus \mathcal{V}_2 &\stackrel{\text{def}}{=} \{\iota_i(V) \mid V \in \mathcal{V}_i\} \\ !\mathcal{S} &\stackrel{\text{def}}{=} \{!S \mid S \in \mathcal{S}\} \\ \mathcal{V} \bullet \mathcal{S} &\stackrel{\text{def}}{=} \{V \bullet S \mid V \in \mathcal{V} \text{ and } S \in \mathcal{S}\} \\ \mathcal{S}_1 \&\mathcal{S}_2 &\stackrel{\text{def}}{=} \{\pi_i \cdot S \mid S \in \mathcal{S}_i\} \\ \diamond \mathcal{V} &\stackrel{\text{def}}{=} \{\diamond V \mid V \in \mathcal{V}\} \end{aligned}$$

**Lemma 53.** *These maps preserve strong normalisation.*

*Proof.* This follows immediately from Lemma 50.  $\square$

**Definition 54.** We define by mutual induction interpretations of types:

$$\begin{array}{ll} \|P\| \subseteq \mathbb{V} & \|N\| \subseteq \mathbb{S} \\ \mathbb{T}(A) \subseteq \mathbb{T} & \mathbb{E}(A) \subseteq \mathbb{E} \end{array}$$

with:

$$\begin{array}{ll} \mathbb{T}(P) \stackrel{\text{def}}{=} \|P\|^{\perp_+ \perp_+} & \mathbb{T}(N) \stackrel{\text{def}}{=} \|N\|^{\perp_-} \\ \mathbb{E}(P) \stackrel{\text{def}}{=} \|P\|^{\perp_+} & \mathbb{E}(N) \stackrel{\text{def}}{=} \|N\|^{\perp_- \perp_-} \\ \|X^+\| \stackrel{\text{def}}{=} \mathbb{V} & \|X^-\| \stackrel{\text{def}}{=} \mathbb{S} \\ \|\mathbf{1}\| \stackrel{\text{def}}{=} \{()\} & \|A \multimap B\| \stackrel{\text{def}}{=} \mathbb{T}(A)_{\mathbb{V}} \bullet \mathbb{E}(B)_{\mathbb{S}} \\ \|A \otimes B\| \stackrel{\text{def}}{=} \mathbb{T}(A)_{\mathbb{V}} \otimes \mathbb{T}(B)_{\mathbb{V}} & \|A \& B\| \stackrel{\text{def}}{=} \mathbb{E}(A)_{\mathbb{S}} \& \mathbb{E}(B)_{\mathbb{S}} \\ \|\mathbf{!}A\| \stackrel{\text{def}}{=} (\mathbb{E}(A)_{\mathbb{S}})^{\perp_+}_{\mathbb{V}} & \|\Diamond A\| \stackrel{\text{def}}{=} (\Diamond \mathbb{T}(A)_{\mathbb{V}})^{\perp_-}_{\mathbb{S}} \\ \|A \oplus B\| \stackrel{\text{def}}{=} \mathbb{T}(A)_{\mathbb{V}} \oplus \mathbb{T}(B)_{\mathbb{V}} & \\ \|\mathbf{0}\| \stackrel{\text{def}}{=} \emptyset & \|\top\| \stackrel{\text{def}}{=} \emptyset \end{array}$$

From now on it will be clear from the polarity of the formula whether  $\cdot^{\perp}$  means  $\cdot^{\perp_+}$  or  $\cdot^{\perp_-}$ .

**Lemma 55.** For all  $A$  one has:

$$\mathbb{T}(A) = \mathbb{E}(A)_{\mathbb{S}}^{\perp} \quad \mathbb{E}(A) = \mathbb{T}(A)_{\mathbb{V}}^{\perp}.$$

*Proof.* If  $A$  is positive, then  $\mathbb{E}(A_+)_{\mathbb{S}} = \mathbb{E}(A_+)$  and  $\mathbb{T}(A) = \mathbb{E}(A)^{\perp}$  by definition; if it is negative then we prove  $\mathbb{T}(A_-) = \mathbb{E}(A_-)_{\mathbb{S}}^{\perp}$  by inclusion.  $(\supseteq)$  One has  $\|A_-\| \subseteq \mathbb{E}(A_-)_{\mathbb{S}}$ , hence  $\mathbb{T}(A_-) = \|A_-\|^{\perp} \supseteq \mathbb{E}(A_-)_{\mathbb{S}}^{\perp}$ .  $(\subseteq)$  One has  $\mathbb{E}(A_-)_{\mathbb{S}} \subseteq \mathbb{E}(A_-) = \|A_-\|^{\perp_+}$ , hence  $\mathbb{T}(A_-) = \|A_-\|^{\perp} \subseteq \mathbb{E}(A_-)_{\mathbb{S}}^{\perp}$ . The reasoning for  $\mathbb{E}(A) = \mathbb{T}(A)_{\mathbb{V}}^{\perp}$  is symmetric.  $\square$

**Lemma 56.** Let  $A$  be a formula. For all  $x$  one has  $x \in \mathbb{T}(A)_{\mathbb{V}}$  and for all  $\alpha$  one has  $\alpha \in \mathbb{E}(A)_{\mathbb{S}}$ .

*Proof.* This is immediate from the definitions of  $\mathbb{T}$  and  $\mathbb{E}$ .  $\square$

**Definition 57.** For  $\Gamma = (x_1 : A_1, \dots, x_n : A_n)$  and  $\Delta = (\alpha : B)$  (possibly empty), and any substitution  $\sigma : \{x_1, \dots, x_n, \alpha\} \rightarrow \mathbb{V} \cup \mathbb{S}$ , we write  $\sigma \Vdash \Gamma, \Delta$  whenever  $\forall i, \sigma(x_i) \in \mathbb{T}(A_i)_{\mathbb{V}}$  and  $\sigma(\alpha) \in \mathbb{E}(A)_{\mathbb{S}}$ .

**Theorem 58.** Consider  $\sigma \Vdash \Gamma, \Delta$ . One has:

- if  $c : (\Gamma \vdash \Delta)$  then  $c[\sigma] \in \perp\!\!\!\perp$ ,
- if  $\Gamma \vdash t : A$  then  $t[\sigma] \in \mathbb{T}(A)$ ,
- if  $\Gamma \mid e : A \vdash \Delta$  then  $e[\sigma] \in \mathbb{E}(A)$ .

*Proof.* This is proved by induction on the derivations.

*Rules  $(\vdash ax)$  and  $(ax \vdash)$ :* immediate. *Rule  $(\vdash \mu^\varepsilon)$ :* One has to show  $\mu\alpha^\varepsilon.c[\sigma] \in \mathbb{T}(A_\varepsilon)$  for some  $\sigma \Vdash \Gamma$  with  $\alpha \notin \text{dom } \Gamma$ . By Lemma 55, it suffices to show  $\langle \mu\alpha^\varepsilon.c[\sigma] \parallel S \rangle^\varepsilon \in \perp\!\!\!\perp$  for all  $S \in \mathbb{E}(A_\varepsilon)_{\mathbb{S}}$ . By Proposition 51, this follows from  $c[\sigma, S/\alpha] \in \perp\!\!\!\perp$  which follows from the induction hypothesis. *Rule  $(\bar{\mu}^\varepsilon \vdash)$ :* same reasoning.

*Rule (cut):* One has to show  $\langle t \parallel e \rangle^\varepsilon[\sigma] \in \perp\!\!\!\perp$  for any  $\sigma \Vdash \Gamma, \Gamma', \Delta$ . We consider the restrictions  $\sigma' \Vdash \Gamma$  and  $\sigma'' \Vdash \Gamma', \Delta$  of  $\sigma$ . By induction hypothesis, one has  $t[\sigma'] \in \mathbb{T}(A_\varepsilon)$  and  $e[\sigma''] \in \mathbb{E}(A_\varepsilon)$ , hence the result from  $\mathbb{T}(A_\varepsilon) \perp\!\!\!\perp^\varepsilon \mathbb{E}(A_\varepsilon)$ .

*Rules ( $\sigma$ ), ( $\vdash_f \sigma$ ) and ( $\sigma \vdash$ )* for  $\sigma \in \Sigma^1(\Gamma; \Gamma')$ . For any  $\sigma' \Vdash \Gamma'$  one has  $(\sigma' \circ \sigma) \Vdash \Gamma$ , from which the result follows by induction.

*Rule ( $\rightarrow_f$ ):* One has to show  $V \cdot S[\sigma] \in \mathbb{T}(A)_\vee \bullet \mathbb{E}(B)_\text{s}$  for any  $\sigma \Vdash \Gamma, \Gamma', \Delta$ . We consider the restrictions  $\sigma' \Vdash \Gamma$  and  $\sigma'' \Vdash \Gamma', \Delta$  of  $\sigma$ . By induction, one has  $V[\sigma'] \in \mathbb{T}(A)_\vee$  and  $S[\sigma''] \in \mathbb{E}(B)_\text{s}$ , hence the result. *Same reasoning for the rules ( $\vdash_f \otimes$ ), ( $\&_i \vdash_f$ ), and ( $\vdash_f \oplus_i$ ).*

*Rule ( $\vdash_f \&$ ):* the goal is to prove  $\mu \langle \alpha.c; \beta.c' \rangle \in \mathbb{T}(A \& B)$  for  $\sigma \Vdash \Gamma$  with  $\alpha, \beta \notin \text{dom } \Gamma$ . By definition it is enough to prove  $\langle \mu \langle \alpha.c; \beta.c' \rangle [\sigma] \parallel \pi_i \cdot S_i \rangle^- \in \perp\!\!\!\perp$  for both  $S_1 \in \mathbb{E}(A)_\text{s}$  and  $S_2 \in \mathbb{E}(B)_\text{s}$ . By Proposition 51, this follows from  $\mu \langle \alpha.c; \beta.c' \rangle [\sigma] \in \mathbb{T}$ ,  $c[\sigma, S_1/\alpha] \in \perp\!\!\!\perp$  and  $c'[\sigma, S_2/\beta] \in \perp\!\!\!\perp$ . The latter two follow from induction hypothesis. The former follows from Lemma 50 and  $c[\sigma], c'[\sigma] \in \perp\!\!\!\perp$  obtained by the induction hypothesis by substituting  $\alpha$  and  $\beta$  by themselves by Lemma 56. *Same reasoning for the rules ( $\vdash_f \rightarrow$ ), ( $\otimes \vdash$ ), ( $\oplus \vdash$ ).*

*Rule ( $\vdash_f \top$ ):* one has to show  $\mu \langle V \rangle [\sigma] \in \emptyset^{\perp\!\!\!\perp} = \mathbb{V}$ . By Lemma 50, this follows from  $V[\sigma] \in \mathbb{V}$  which follows by the induction hypothesis. *Same reasoning for the rule ( $\mathbf{0} \vdash_f$ ).*

*Rule ( $\vdash_f !$ ):* we show  $\mu !\alpha.c[\sigma] \in \parallel !A \parallel = (!\mathbb{E}(A)_\text{s})^{\perp\!\!\!\perp}_\vee$ . We show for any  $S \in \mathbb{E}(A)_\text{s}$ ,  $\langle \mu !\alpha.c[\sigma] \parallel !S \rangle^+ \in \perp\!\!\!\perp$ . By Proposition 51 this follows from  $c[\sigma, S/\alpha] \in \perp\!\!\!\perp$  which follows from induction hypothesis. *Same reasoning for the rule ( $\diamond \vdash$ ).*

*Rule ( $! \vdash_f$ ):* the goal is to show  $!S[\sigma] \in (!\mathbb{E}(A)_\text{s})^{\perp\!\!\!\perp}_\vee$ . One has  $!S[\sigma] \in !\mathbb{E}(A)_\text{s}$  by induction hypothesis, and  $!\mathbb{E}(A)_\text{s} \subseteq (!\mathbb{E}(A)_\text{s})^{\perp\!\!\!\perp}_\vee$  follows from  $(!\mathbb{E}(A)_\text{s})^{\perp\!\!\!\perp}_\vee \subseteq !\mathbb{E}(A)_\text{s}$ . *Same reasoning for the rule ( $\vdash_f \diamond$ ).*  $\square$

**Theorem 59** (Strong normalization). *Any typable term is strongly normalizing.*

*Proof.* For  $c : (\Gamma \vdash \Delta)$ ,  $\Gamma \vdash t : A$  and  $\Gamma \mid e : A \vdash \Delta$ , we show  $c \in \perp\!\!\!\perp$ ,  $t \in \mathbb{T}$  and  $e \in \mathbb{E}$ . This follows from Lemma 58 applied with the identity substitution, which indeed satisfies  $\text{id} \Vdash \Gamma, \Delta$  by Lemma 56.  $\square$

## E.2. Completeness of focusing

We describe focused normal forms (Andreoli, 1992; Laurent, 2004; Liang and Miller, 2009) and prove their completeness, strengthened with respect to the interpretation: for any proof there is an *equivalent* focused proof. This section is taken from Munch-Maccagnoni (2017) with slight adaptations.

In this section,

- sequents are measured by the number of connectives and units,
- terms are measured by the number of subterms of the form  $\langle V \parallel \alpha \rangle^+$ ,  $\langle x \parallel S \rangle^-$ ,  $!S$  or  $\diamond V$  in their  $\rightarrow_R$ -normal form if it exists (in which case it is unique by confluence), otherwise it is infinite by convention.

**Definition 60.** We define an *inversion* relation  $>$  between sequents and multisets of sequents:

$$\begin{aligned}
(\Gamma \vdash A \multimap B) &> \{(\Gamma, A \vdash B)\} \\
(\Gamma, A \otimes B, \Gamma' \vdash \Delta) &> \{(\Gamma, A, B, \Gamma' \vdash \Delta)\} \\
(\Gamma, \mathbf{1}, \Gamma' \vdash \Delta) &> \{(\Gamma, \Gamma' \vdash \Delta)\} \\
(\Gamma \vdash A \& B) &> \{(\Gamma \vdash A), (\Gamma \vdash B)\} \\
(\Gamma, A \oplus B, \Gamma' \vdash \Delta) &> \{(\Gamma, A, \Gamma' \vdash \Delta), (\Gamma, B, \Gamma' \vdash \Delta)\} \\
(\Gamma \vdash \top) &> \emptyset \\
(\Gamma, \mathbf{0}, \Gamma' \vdash \Delta) &> \emptyset
\end{aligned}$$

A sequent that is normal for  $>$  is called *inverted*.

In other words, a sequent  $\Gamma \vdash \Delta$  is inverted if:

- $\Gamma$  only contains formulae that are either negative or of the form  $!A$  or  $X^+$ ; and
- $\Delta$  contains either a positive formula, or a formula of the form  $X^-$  or  $\Diamond A$ .

**Lemma 61.** *The extension of  $>$  to a relation between multisets of sequents is terminating and confluent.*

*Proof.* The relation is strictly decreasing for the induced multiset order, therefore it is terminating. It is confluent by Newman's lemma because it is locally confluent.  $\square$

**Lemma 62.** *Let  $f$  be any term. Then  $f[x \otimes y/z]$ ,  $f[\iota_i(x)/y]$ ,  $f[(\cdot)/x]$ ,  $f[x \cdot \alpha/\beta]$ , and  $f[\pi_i \cdot \alpha/\beta]$  are smaller than  $f$ .*

*Proof.* Note that if  $f$  is not  $\rightarrow_R$ -normalisable then the result is trivial by convention. For any term  $f$  let us write  $|f|$  the number of subterms of the form  $\langle V \parallel \alpha \rangle^+$ ,  $\langle x \parallel S \rangle^-$ ,  $!S$  or  $\Diamond V$ , and  $f \downarrow$  the  $\rightarrow_R$ -normal form of  $f$  if it exists. In particular, if  $f$  is normalisable then its size is given by  $|f \downarrow|$ . We now consider  $f[p/\kappa]$  as in the statement. With these notations the result is stated  $|f[p/\kappa] \downarrow| \leq |f \downarrow|$ .

We first prove the result for  $f \rightarrow_R$ -normal, that is  $|f[p/\kappa] \downarrow| \leq |f|$ . Since  $f$  is  $\rightarrow_R$ -normal, the only redexes in  $f[p/\kappa]$  are the ones created by the substitution. Necessarily, those are of the form  $\langle \mu p.C \parallel p \rangle^-$  or  $\langle p \parallel \bar{\mu} p.C \rangle^+$  (up to renaming of bound variables), and the reduction just selects a field of  $C$ , thus decreasing  $|\cdot|$  without creating new redexes. Iterating the argument by induction on  $f[p/\kappa]$  to eliminate all the redexes in this way, we obtain its  $\rightarrow_R$ -normal form<sup>6</sup> and it verifies  $|f[p/\kappa] \downarrow| \leq |f[p/\kappa]|$ . We conclude by noting  $|f[p/\kappa]| \leq |f|$  since substitution can only decrease the number of commands of the form  $\langle V \parallel \alpha \rangle^+$  or  $\langle x \parallel S \rangle^-$ .

Now, in the case of  $f \rightarrow_R$ -normalisable, we have just proved that  $(f \downarrow[p/\kappa]) \downarrow$  exists and  $|(f \downarrow[p/\kappa]) \downarrow| \leq |f \downarrow|$ . But  $(f \downarrow[p/\kappa]) \downarrow = f[p/\kappa] \downarrow$  via the reduction  $f[p/\kappa] \rightarrow_R^* f \downarrow[p/\kappa] \rightarrow_R^* (f \downarrow[p/\kappa]) \downarrow$  and uniqueness of the normal form.  $\square$

**Lemma 63.** *For any  $c : \Psi$ , and for any  $\{\Psi_1, \dots, \Psi_n\} < \Psi$ , there exists an equivalent command  $c' : \Psi$  derived from smaller commands of type  $\Psi_1, \dots, \Psi_n$  by application of a rule in Fig. 7.*

<sup>6</sup>It is in fact the parallel reduct of the redexes created by the substitution, and also the complete development of the substitution, but it is not necessary to use these notions here.

$$\begin{array}{c}
\frac{c : (\Gamma, x : A \vdash \alpha : B)}{\langle \mu(x \cdot \alpha).c \parallel \beta \rangle : (\Gamma \vdash \beta : A \multimap B)} \\
\frac{c : (\Gamma, x : A, y : B, \Gamma' \vdash \Delta)}{\langle z \parallel \bar{\mu}(x \otimes y).c \rangle : (\Gamma, z : A \otimes B, \Gamma' \vdash \Delta)} \\
\frac{c : (\Gamma, \Gamma' \vdash \Delta)}{\langle x \parallel \bar{\mu}().c \rangle : (\Gamma, x : \mathbf{1}, \Gamma' \vdash \Delta)} \\
\frac{c : (\Gamma \vdash \alpha : A) \quad c' : (\Gamma \vdash \beta : B)}{\langle \mu < \alpha.c; \beta.c' > \parallel \gamma \rangle : (\Gamma \vdash \gamma : A \& B)} \\
\frac{c : (\Gamma, x : A, \Gamma' \vdash \Delta) \quad c' : (\Gamma, y : B, \Gamma' \vdash \Delta)}{\langle z \parallel \bar{\mu}[x.c \mid y.c'] \rangle : (\Gamma, z : A \oplus B, \Gamma' \vdash \Delta)} \\
\frac{}{\langle x \parallel \bar{\mu}[\ ]_{\Gamma, \Gamma', \Delta} \rangle : (\Gamma, x : \mathbf{0}, \Gamma' \vdash \Delta)} \\
\frac{}{\langle \mu < >_{\Gamma} \parallel \alpha \rangle : (\Gamma \vdash \alpha : \top)}
\end{array}$$


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Figure 7: Inversion

$$\begin{array}{c}
\frac{c : (\Gamma \vdash \alpha : N)}{\Gamma \vdash \mu \alpha^-.c : N \mid} \quad \frac{c : (\Gamma, x : P \vdash \Delta)}{\Gamma \mid \bar{\mu} x^+.c : P \vdash \Delta} \quad \frac{c : (!\Gamma \vdash \alpha : A)}{!\Gamma \vdash \mu !\alpha.c : !A \mid} \quad \frac{c : (!\Gamma, x : A \vdash \Diamond \Delta)}{!\Gamma \mid \bar{\mu} \Diamond x.c : \Diamond A \vdash \Diamond \Delta} \\
\frac{}{!\Gamma, x : X^+ \vdash x : X^+ \mid} \quad \frac{}{!\Gamma, x : !A, !\Gamma' \vdash x : !A \mid} \quad \frac{}{!\Gamma \mid \alpha : X^- \vdash \alpha : X^-} \\
\frac{!\Gamma', \Gamma_{\vdash \mathbf{fv} V} \vdash V : A \mid \quad !\Gamma', \Gamma_{\vdash \mathbf{fv} W} \vdash W : B \mid}{!\Gamma', \Gamma \vdash V \otimes W : A \otimes B \mid} \quad \frac{!\Gamma', \Gamma_{\vdash \mathbf{fv} V} \vdash V : A \mid \quad !\Gamma', \Gamma_{\vdash \mathbf{fv} S} \mid S : A \vdash \Delta}{!\Gamma', \Gamma \mid V \cdot S : A \multimap B \vdash \Delta} \\
\frac{}{!\Gamma \vdash () : \mathbf{1} \mid} \quad \frac{\Gamma \vdash V : A_i \mid}{\Gamma \vdash \iota_i(V) : A_1 \oplus A_2 \mid} \quad \frac{\Gamma \mid S : A_i \vdash \Delta}{\Gamma \mid \pi_i \cdot S : A_1 \& A_2 \vdash \Delta}
\end{array}$$


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Figure 8: Focusing in  $\mathbf{ILL}_p^\Diamond$

*Proof.* The result follows by mapping each inversion as follows:

$$\begin{aligned}
c : (\Gamma, z : A \otimes B, \Gamma' \vdash \Delta) &> \{ \langle x \otimes y \parallel \bar{\mu}z.c \rangle : (\Gamma, x : A, y : B, \Gamma' \vdash \Delta) \} \\
c : (\Gamma, z : A \oplus B, \Gamma' \vdash \Delta) &> \{ \langle \iota_1(x) \parallel \bar{\mu}z.c \rangle : (\Gamma, x : A, \Gamma' \vdash \Delta), \langle \iota_2(y) \parallel \bar{\mu}z.c \rangle : (\Gamma, y : B, \Gamma' \vdash \Delta) \} \\
c : (\Gamma, x : \mathbf{1}, \Gamma' \vdash \Delta) &> \{ \langle () \parallel \bar{\mu}x.c \rangle : (\Gamma, \Gamma' \vdash \Delta) \} \\
c : (\Gamma \vdash \beta : A \multimap B) &> \{ \langle \mu\beta.c \parallel x.\alpha \rangle : (\Gamma, x : A \vdash \alpha : B) \} \\
c : (\Gamma \vdash \gamma : A \& B) &> \{ \langle \mu\gamma.c \parallel \pi_1.\alpha \rangle : (\Gamma \vdash \alpha : A), \langle \mu\gamma.c \parallel \pi_2.\beta \rangle : (\Gamma \vdash \beta : B) \} \\
c : (\Gamma, x : \mathbf{0}, \Gamma' \vdash \Delta) &> \emptyset \\
c : (\Gamma \vdash \alpha : \top) &> \emptyset
\end{aligned}$$

The commands on the right-hand side are well-typed by straightforward derivation. They are smaller by application of Lemma 62. When composed with the respective constructions of Fig. 7, one obtains a command  $c' \simeq_{\text{RE}} c$  and thus equivalent.  $\square$

**Proposition 64** (Inversion). *For any sequent  $\Psi$ , there exist inverted sequents  $\Psi_1, \dots, \Psi_n$  such that any command  $c : \Psi$  can up to equivalence be derived from smaller commands  $c_1 : \Psi_1, \dots, c_n : \Psi_n$  by applying the rules in Fig. 7 in any order.*

*Proof.* By Lemmas 61 and 63.  $\square$

**Proposition 65** (Focusing). *For any typable command  $c : (\Gamma \vdash \Delta)$  where  $\Gamma \vdash \Delta$  is inverted, there exist equivalent command and derivation of either of the following forms:*

$$\frac{! \Gamma', \Gamma'' \vdash V : \Delta(\alpha)_+}{\langle V \parallel \alpha \rangle^+ : (\Gamma \vdash \Delta)} \quad \frac{! \Gamma', \Gamma'' \mid S : \Gamma(x)_- \vdash \Delta}{\langle x \parallel S \rangle^- : (\Gamma \vdash \Delta)} \quad \frac{! \Gamma', \Gamma'' \mid S : \Gamma'(x) \vdash \Delta}{\langle x \parallel !S \rangle^+ : (\Gamma \vdash \Delta)} \quad \frac{! \Gamma', \Gamma'' \vdash V : \Delta(\alpha)}{\langle \diamond V \parallel \alpha \rangle^- : (\Gamma \vdash \Delta)}$$

where  $! \Gamma'$  is the restriction of  $\Gamma$  to formulae of the form  $!A$ , and  $\Gamma''$  is obtained from  $\Gamma$  by removing  $! \Gamma'$  and  $x$  (if applicable). Moreover,  $S$  and  $V$  derive from zero or more commands strictly smaller than  $c$  by applications of rules in Fig. 8.

*Proof.* Up to equivalence, one can assume  $c \rightarrow_{\text{R-normal}}$ : indeed, the normal form exists by Theorem 59, its derivation is equivalent by soundness, and it has the same size by definition. Then one has either  $c = \langle x \parallel S^{\Gamma(x)} \rangle$  or  $c = \langle V^{\Delta(\alpha)} \parallel \alpha \rangle$  by generation lemma, with  $S$  and  $V \rightarrow_{\text{R-normal}}$ . We sort them into one of three cases: either  $\langle V' \parallel \alpha \rangle^+$ ,  $\langle x \parallel S' \rangle^-$ , or  $\langle x \parallel !S' \rangle^+$ , derived as in the above statement. If  $\Gamma(x)$  is negative or  $\Delta(\alpha)$  is positive, we are in one of the first two cases. Since  $\Gamma \vdash \Delta$  is inverted, only three cases remain:  $\Gamma(x) = X^+$ ,  $\Gamma(x) = !A$ ,  $\Delta(\alpha) = X^-$ , and  $\Delta(\alpha) = \diamond A$ . By generation lemma, one respectively has  $S = \alpha$  (first case),  $S = !S'$  (third case),  $V = x$  (second case), and  $V = \diamond V'$  (fourth case). Notice that  $V'$  and  $S'$  are strictly smaller than  $c$ . Then the result follows by an induction on  $S'$  and  $V'$  typable in an inverted sequent, by generation lemma, and applying weakening on the hypotheses. The base cases are

$$\Gamma \mid \alpha : N \vdash \Delta \quad \Gamma \vdash x : P \mid \quad ! \Gamma \vdash \mu! \alpha.c : !A \mid \quad ! \Gamma \mid \bar{\mu} \diamond x.c : \diamond A \vdash \diamond \Delta \quad \Gamma \mid S_0 : P \vdash \Delta \quad \Gamma \vdash V_0 : N \mid$$

In the first two cases, one has indeed  $P = \Gamma(x)$  of the form  $X^+$  or  $!A$  and  $N = \Delta(\alpha)$  of the form  $X^-$  or  $\diamond A$  by inversion. In the last three cases,  $\mu! \alpha.c$ ,  $\bar{\mu} \diamond x.c$ ,  $V_0$ , and  $S_0$  are strictly smaller than  $c$  by being subterms of  $V'$  or  $S'$ . Then in the last two cases we apply a  $\bar{\mu}^+$ - or a  $\mu^-$ -expansion to obtain equivalent terms  $\bar{\mu}x^+.c'$  and  $\mu\alpha^-.c'$ . The expansion preserves the size, so  $c'$  is strictly smaller than  $c$ .  $\square$

Thus, for any proof, there is an equivalent proof obtained by alternating one inversion phase with one or more<sup>7</sup> focusing phases.

<sup>7</sup>in the case where a rule ( $\vdash !$ ) or ( $\diamond \vdash$ ) comes from a hypothesis that is already inverted.